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SUMMARY

A method is given for solving problems associated with Laplace and Poisson equations which, in general, requires considerably fewer equations than the usual methods and which gives a convergent solution by the method of successive approximations. For infinite regions, by this method, the exact solution for the Dirichlet and Neumann problems can be found by solving a system of equations with as many variables as there are boundary points of the region. In addition, at each stage of the iteration a best possible estimate of the error of the approximate solution with respect to the exact solution of the difference equation for the Dirichlet problem is furnished, and, for the Neumann problem, a bound for the error of the normal difference of the approximate solution is given.

INTRODUCTION

Certain problems in steady-state heat flow, gas dynamics, both for compressible and incompressible flows, plane torsion, and so forth can be formulated as problems associated with the Laplace or Poisson equations in two dimensions. A frequently used method of approximating the solution of the Laplace equation consists of replacing the region by those points inside the region or on the boundary whose coordinates are multiples of a fixed positive number, which is the mesh size, and replacing the Laplace equation by the Laplace difference equation which says that the value of the function at a point not on the boundary is the mean of its values at the four neighboring points. This gives a system of as many equations as there are points inside the region. These equations are solved by relaxation or iteration. The present report concerns a formulation of a complete system of equations for as many parameters as there are boundary points of the region where the desired function is a given linear function of these parameters for the Dirichlet and Neumann problems. For example, for a "square region" containing, say, 900 inner points, the number of variables and equations required by the present method is 120. In any case, the larger the region the

greater the utility of the method. This system of equations is given in a form which allows the application of the method of successive approximations, that is, each parameter is given by a linear function of all the parameters. In addition, the method of successive approximations applied to this system of equations gives a solution which is known to converge at least geometrically for all regions. This method is a finite-difference analog of the integral-equation method of potential theory and is based on some results of Courant, Friedrichs, and Lewy (ref. 1), McCrea and Whipple (ref. 2), Stöhr (ref. 3), and Duffin (ref. 4).

The method has the disadvantage that the calculations required for setting up the system of simultaneous equations are much more complex than in the usual methods. However, for some problems the method can be used to reduce the number of variables so that the problem is within the capacity of an automatic computing machine. Also, if more than one problem is to be solved for a given region this method offers a considerable advantage.

Since, as is shown in this report, the Poisson difference equation for a finite region can readily be reduced to the Laplace equation by a simple computation, the above remarks apply also to the Poisson difference equation.

For nonlinear equations which can be written in a form such that the Laplacian of the unknown function equals a given function whose arguments are the unknown function, its derivatives, and the space variables, the method of successive approximations is used, each step consisting of the solution of a Poisson difference equation as, for example, in the Rayleigh-Jansen method.

Another advantage of the present method for the Dirichlet problem is that at any stage a precise bound is automatically provided for the error of the previous approximation and a bound is given for the corresponding error for the Neumann problem of the normal derivative.

On the theoretical side the structure of the functions satisfying the Laplace difference equation can be completely described in terms of discrete potentials, and theorems which extend Duffin's results (ref. 4) can be derived.

In addition, the method of the present report can be applied to the biharmonic difference equation, conformal mapping, and the theory of monodiffric functions (ref. 5). An interpretation in electrical network terms can be used to investigate electrical analog methods for solving the Dirichlet and Neumann problems. Finally, the results of this paper can be extended to n dimensions.

In the present report the equations of the method will be derived and formulated for computational purposes in the analysis section. Worked

examples of the Dirichlet and Neumann methods are discussed. In appendix A the proof of Green's first, second, and third identities is given. In appendixes B and C the homogeneous integral-equation analogs are given for the Neumann and Dirichlet problems, respectively. For those readers who are not familiar with classical potential theory, of which the method presented herein is an analog, a brief sketch is given in appendix D.

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SYMBOLS

A (A_{rs}) matrix

A' (A_{rs}') matrix

$A_{rs} = \varphi(a_{rs})$

$A_{rs}' = \varphi(a_{rs}')$

$A_{rs}'' = \varphi(a_{rk}'')$

$a_{rk}'' = b'' - c''$

$(a_{rs}) = \alpha_{\gamma\gamma} - \alpha_{\gamma\gamma t}$

$a_{rs}' = b' - c'$

B (B_{rs}) matrix

$B_{rs} = \varphi(b_{rs})$

b' (b_{rs}') matrix

$b'' = (b_{rs}'')$

$(b_{rs}) = \beta_{\gamma} - \beta_{\gamma}'$

b_{rs}' coordinates of rth point of graph of normal exterior segments

b_{rs}'' coordinates of rth point of graph of inner normal segments

C matrix analogous to kernel of integral equation of potential theory for Dirichlet problem

C' matrix analogous to kernel of integral equation of potential theory for Neumann problem

C'' matrix used in calculation of $\Delta_1 W$, $-\eta_{\Gamma t}\eta A$

c' $\xi \times \gamma$ matrix whose rows are lists of coordinates of boundary points

c'' $(\gamma + \nu) \times \gamma$ matrix whose rows are lists of coordinates of boundary points

D diagonal matrix whose diagonal entries give number of adjacent inner normal points

G a region, that is, a set consisting of only inner and boundary points

I set of inner points of G

I_γ $\gamma \times \gamma$ identity matrix

$L = L(P,Q) = \varphi(P - Q)$

M $\gamma \times 1$ matrix each of whose rows is dipole magnitude of inner normal segments terminating on boundary point corresponding to given row

m $\gamma \times 1$ matrix whose rows give mass on boundary point corresponding to row for either a simple- or double-layer potential

m' $\gamma \times 1$ matrix whose rows give mass on normal point corresponding to row (double-layer potential)

P point or coordinates of point with integral-valued coordinates

(p_k,q_k) coordinates of kth boundary point of G

(p_k',q_k') coordinates of kth inner normal point of G

(p_k'',q_k'') coordinates of kth point of set of end points of exterior normal segments of G

$\left(p_k''', q_k'''\right)$ coordinates of kth point of set of points consisting of inner normal and boundary points of G

Q point or coordinates of point with integral-valued coordinate

R functional operator which for given point gives value of function on adjacent inner points of G minus product of number of such points and value of function on given point

r distance from origin

s_i ith inner normal segment

s_i' ith exterior normal segment

U function defined on G which satisfies Laplace difference equation on inner points

$\bar{U}$ $\gamma \times 1$ matrix whose elements are values of U on boundary points

V function defined on G

W potential of simple- or double-layer distribution

$\bar{W}$ $\gamma \times 1$ matrix of values of W on boundary points

$\bar{W}''$ $\xi \times 1$ matrix of values of simple-layer potential W on end points of exterior normal segments

$\bar{W}'''$ $(\nu + \gamma) \times 1$ matrix of values of W on normal inner and boundary points

(x,y) integral-valued coordinates

$\alpha_{\gamma\gamma}$ $\gamma \times \gamma$ matrix where each column is list of coordinates of boundary points

β number of inner normal segments

γ number of boundary points

γ_k diagonal elements of D_Γ

$\Delta_1\bar{U}, \Delta_1\bar{W}, \Delta_e\bar{W}$ factors of $\gamma \times 1$ matrix

δ	difference operator
ϵ^k	$\gamma \times 1$ matrix of errors of kth approximation of U
η	incidence matrix of graph of inner normal segments
η_N	columns of η corresponding to normal inner points of G
η_Γ	columns of η corresponding to boundary points of G
η'	incidence matrix of graph of exterior normal segments
η''	columns of η' corresponding to boundary points
θ	angle in polar coordinates
μ	magnitude of dipole for oriented unit segment
ν	number of normal points
ξ	number of points in graph of exterior normal segments
σ	matrix whose rows correspond to normal points and which indicates boundary points adjacent to each normal point
τ	number of exterior normal segments
φ	fundamental solution of Laplace difference equation
Ω	constant in bound for asymptotic expression of φ
ω	Euler's constant

Superscript:

k	denotes kth approximation of given quantity

Subscripts:

k	denotes kth approximation of given quantity
r,s	integral values
t	transpose
Γ	set of boundary points of G

ANALYSIS

Geometrical Definitions

The set of points in the plane whose coordinates are integers will be considered. Two points will be said to be adjacent if the distance between them is unity. A subset of points G will be called connected if for any two points of G there is a path consisting of segments of unit length connecting them in such a manner that the end points of these segments are all in G. A point of G is an inner point if it is adjacent to four points of G. A point of G is a boundary point if it is not an inner point but is adjacent to an inner point. A point of G is an inner normal point if it is an inner point and is adjacent to a boundary point. A unit segment bounded by one inner normal point and one boundary point is an inner normal segment of G. A set of points is a region if it is connected and every point of the set is either an inner or a boundary point. The symbol G will be used in the following discussion to denote a region. The coordinates of the inner points will be denoted by (x,y); the coordinates of the inner normal points, by (p_k',q_k') where $k = 1, 2, \ldots \gamma$; the coordinates of the boundary points, by (p_k,q_k) where $k = 1, 2, \ldots \nu$; and the inner normal segments, by s_k where $k = 1, 2, \ldots \beta$.

The matrices $\eta_N = \left(\eta_{Nik}\right)$ and $\eta_\Gamma = \left(\eta_{\Gamma ik}\right)$ are defined by

$$\eta_{Nik} = \begin{cases} -1 & \text{if } \left(p_k',q_k'\right) \text{ is an end point of } s_i \\ 0 & \text{if } \left(p_k',q_k'\right) \text{ is not an end point of } s_i \end{cases} \tag{1}$$

$$\eta_{\Gamma ik} = \begin{cases} 1 & \text{if } \left(p_k,q_k\right) \text{ is an end point of } s_i \\ 0 & \text{if } \left(p_k,q_k\right) \text{ is not an end point of } s_i \end{cases} \tag{2}$$

It will be assumed that the boundary points of the regions considered consist of a finite set of points.

Discrete Potentials and Poisson Difference Equation

Following the notation of Stöhr (ref. 3), the fundamental solution of the Laplace difference equation in two dimensions is written as

$$\varphi(x,y) = \varphi(P)$$

where x and y are integers and $\varphi(P)$ has the following properties:

$$\varphi(x,y) = \varphi(|x|,|y|) = \varphi(|y|,|x|) \tag{3}$$

$$R\left[\varphi(P)\right] = \left\{\begin{array}{l} 1 \text{ if } P \text{ is origin} \\ 0 \text{ if } P \text{ is not origin} \end{array}\right\} \tag{4}$$

where

$$R\left[\varphi(x,y)\right] = \varphi(x+1,y) + \varphi(x-1,y) + \varphi(x,y+1) + \varphi(x,y-1) - 4\varphi(x,y) \tag{5}$$

Table I gives the values of $\varphi(p,q)$. In addition, the following asymptotic estimate is given by Stöhr. There is a positive constant Ω such that

$$\left| \varphi(x,y) - \left(\frac{3}{4\pi}\log_e{}^2 + \frac{1}{2\pi}\omega\right) - \frac{1}{2\pi}\log_e r \right| \leqq \frac{\Omega}{r^2} \tag{6}$$

where

$$r = \sqrt{x^2 + y^2}$$

$$\omega = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log_e n\right)$$

In the present paper this function will be used to define a function of four variables. If $P = (x_1,y_1)$ and $Q = (x_2,y_2)$ then $P - Q = \left[(x_1 - x_2),(y_1 - y_2)\right]$ and $L(P,Q)$ is defined by

$$L(P,Q) = \varphi(P - Q) = \varphi\left[(x_1 - x_2),(y_1 - y_2)\right] \tag{7}$$

By equations (3) and (4)

$$L(P,Q) = \varphi\Big[(x_1 - x_2),(y_1 - y_2)\Big] = \varphi\Big(|x_1 - x_2|,|y_1 - y_2|\Big) = L(Q,P) \tag{8}$$

and

$$R_Q\big[L(P,Q)\big] = R_P\big[L(P,Q)\big] = \begin{cases} 1 & \text{if } P = Q \\ 0 & \text{if } P \neq Q \end{cases} \tag{9}$$

where R_Q means that (x,y) in equation (5) is to be taken as (x_2,y_2) and R_P means that (x,y) in equation (5) is to be taken as (x_1,y_1).

If n arbitrary points (x_1,y_1), (x_2,y_2), . . . (x_n,y_n) are given and if to the kth point (x_k,y_k) there corresponds a real number m_k where $k = 1, 2, \ldots n$, then the function

$$W(P) = \sum_{k=1}^{n} m_k L(P,Q_k) \tag{10}$$

where $Q_k = (x_k,y_k)$ will be called the potential of the mass distribution whose density is $m_k = m(x_k,y_k)$ on the given points and zero elsewhere. By equation (9)

$$R\big[W(P)\big] = \begin{cases} m_k & \text{if } P = Q_k \\ 0 & \text{if } P \neq Q_k \end{cases} \tag{11}$$

where $k = 1, 2, \ldots n$.

If $f(P)$ is a function defined on the inner points of a region then the Poisson difference equation is

$$R\big[V(P)\big] = f(P) \tag{12}$$

for each inner point of the region, where $V(P)$ may be subject to additional conditions at boundary points and normal points. If $U(P)$ is defined by

$$U(P) = V(P) - \sum_{k} f(Q_k)L(P,Q_k) \tag{13}$$

where the sum is to be taken over all inner points $Q_1, Q_2, \ldots$, then by equations (11) and (12),

$$R[U(P)] = 0 \tag{14}$$

at all inner points. The conditions on $U(P)$ at the boundary points, or at the boundary and normal points, can be computed from the corresponding conditions on $V(P)$ and equation (13). In this way a problem for the Poisson difference equation can be reduced to a problem for the Laplace difference equation by a direct calculation provided that the region is finite. For an infinite region the above reduction requires further study relating to the convergence of approximations for the sum in equation (13).

Simple- and Double-Layer Potentials

A simple-layer potential is the potential of a mass distribution on the boundary points of a region and is given by

$$W(x,y) = W(P) = \sum_{k=1}^{\gamma} m_k L(P,Q_k) = \sum_{k=1}^{\gamma} m_k \varphi\left[(x - p_k),(y - q_k)\right] \tag{15}$$

where m_k is the mass on the kth boundary point $Q_k = (p_k, q_k)$. Since there are no masses at the inner points, the simple-layer potential satisfies the Laplace difference equation on the inner points. It will be seen later that the solution of the Neumann problem can be represented by such a potential.

A dipole of magnitude μ is defined as a pair of masses situated on the end points of a segment where one of the masses is μ and the other mass is $-\mu$. A double-layer potential is defined as the potential due to a dipole distribution on the normal segments of a region. The following convention will be observed: A dipole magnitude μ_k is associated with the segment $s_k (k = 1, 2, \ldots \beta)$ and the mass of magnitude μ_k associated with s_k is to be considered as being on the boundary point of the segment s_k while the mass of magnitude $-\mu_k$ of this dipole is to be considered as being on the normal point of the segment s_k. It is a consequence of the definitions of η_N and η_T (eqs. (1) and (2)) that

$$m_s' = \sum_{r=1}^{\beta} \eta_{Nrs}\mu_r \tag{16}$$

where $k = 1, 2, \ldots \nu$ and

$$m_s = \sum_{r=1}^{\beta} \eta_{\Gamma rs}\mu_r \tag{17}$$

where $k = 1, 2, \ldots \gamma$ and where m_s and m_s' are the masses associated with the dipole potentials at the boundary and normal points, respectively.

If $m_t' = m_1', m_2', \ldots m_\nu'$, $m_t = m_1, m_2, \ldots m_\gamma$, and $\mu_t = \mu_1, \mu_2, \ldots \mu_\beta$ where the subscript t means that the transpose is to be taken, then the above equations can be written as

$$m' = \eta_{Nt}\mu \tag{18}$$

$$m = \eta_{\Gamma t}\mu \tag{19}$$

Hence the potential $W(P)$ of a double layer is

$$W(x,y) = W(P) = \sum_{k=1}^{\nu} m_k'\varphi\left[(x - p_k'),(y - q_k')\right] + \sum_{k=1}^{\gamma} m_k\varphi\left[(x - p_k),(y - q_k)\right] \tag{20}$$

This potential satisfies the Laplace difference equation at all inner points which are not normal points; at normal points by equation (9),

$$R\left[W\left(p_k', q_k'\right)\right] = m_k' \tag{21}$$

where $k = 1, 2, \ldots \nu$. The double-layer potentials will be restricted by the condition that segments with common boundary points have the same dipole magnitudes. Thus, if the order of the kth boundary point is γ_k (i.e., the number of normal segments on this boundary point is γ_k), then the dipole magnitudes of each of the segments is $(\gamma_k)^{-1}m_k$. Let D_Γ be defined by

$$D_\Gamma = \eta_{\Gamma t}\eta_\Gamma \tag{22}$$

It may be noted that D_Γ is a diagonal matrix whose diagonal elements are precisely the orders of the corresponding boundary points. Thus, in matrix form, the above condition can be written

$$\mu = \eta_\Gamma D_\Gamma^{-1} m \tag{23}$$

and it will be assumed that all double-layer potentials considered comply with this condition. If both sides of equation (23) are multiplied by η_{Nt} and σ is defined by

$$\sigma = -\eta_{Nt}\eta_\Gamma \tag{24}$$

then, by equation (18),

$$m' = -\sigma D_\Gamma^{-1} m \tag{25}$$

The reason for this restriction is that it guarantees the existence of a function $U(P)$ defined on G so that

$$U(P) = W(P) \tag{26}$$

$$R\left[U(P)\right] = 0 \tag{27}$$

for every inner point P belonging to G. To prove this assertion let $\bar{U}$ be defined by

$$\bar{U}_t = (U_1, U_2, \ldots U_\gamma) \tag{28}$$

and let $U(P) = W(P)$ if P is an inner point and $U(p_k, q_k) = U_k$ for boundary points. Since $W(P)$ satisfies the Laplace difference equation at all nonnormal inner points it is only necessary to verify that $\bar{U}$ can be determined so that equation (27) is satisfied on the normal points. By equations (21) and (27) a necessary and sufficient condition that $\bar{U}$ can be determined so that equation (27) is satisfied is that the system of equations

$$R\left[W(p_k', q_k') - U(p_k', q_k')\right] = m_k' \tag{29}$$

where $k = 1, 2, \ldots \nu$, have a solution. Since W and U have identical values at all inner points, equation (29) states that at each normal

inner point the sum of the differences $W - U$ at the adjacent boundary points is equal to the mass at the given normal point. If $\bar{W}$ is defined by

$$\bar{W}_t = \left(W(p_1,q_1),\ W(p_2,q_2),\ \cdots\ W(p_\gamma,q_\gamma)\right) \tag{30}$$

then by the definition of σ (eq. (24)), the system of equations (eq. (29)) can be written

$$\sigma(\bar{W} - \bar{U}) = m' \tag{31}$$

By equation (25) it follows that

$$\bar{W} - \bar{U} = -D_\Gamma^{-1} m \tag{32}$$

is a solution of equation (31) which defines $\bar{U}$. This solution which defines $\bar{U}$ and consequently U will be called the harmonic extension of W on G. Hence, by equation (23)

$$\mu = -\eta_\Gamma(\bar{W} - \bar{U}) \tag{33}$$

Equation (33) is the discrete analog of the discontinuity at the boundary of a double-layer potential of the classical theory, and, as in the classical theory, a double-layer potential will be used to solve the Dirichlet problem.

Reduction of Dirichlet Problem to Integral-Equation Analog

The Dirichlet problem for a bounded region consists of finding a function U defined on a given region which satisfies the Laplace difference equation on the inner points of the region and which assumes arbitrary prescribed values on the boundary points.

Since by equations (18), (19), and (20) $W(P)$ is a linear function of μ, then by equation (30) $\bar{W}$ is a linear function of μ. In equation (33) if $\bar{U}$ is taken as given by the prescribed values of U on the boundary then this system of equations can be regarded as a system of linear equations for μ. If the system of equation (33) has a solution for μ, then m and m' can be calculated by equations (18) and (19), and W can be calculated by equation (20). If $U(P)$ is defined by the condition that it equals $W(P)$ at inner points and, on the boundary points, coincides with the prescribed values of the Dirichlet problem, then U is the solution of the Dirichlet problem. This can be seen by the following argument. If equation (33) is multiplied by η_{Nt} on the left, then equation (31) is a consequence of equation (24). This means

that U is the harmonic extension of W; that is, U satisfies the Laplace difference equation on the inner points of G. Since by the construction U assumes the prescribed values, U(P) is the desired solution and, hence, W represents U on the inner points of G. This result may be formulated in the following way. For a given value of $\bar{U}$, if the system of equations (33) has a solution for μ, then the double-layer potential defined by μ represents U on the inner points of G.

Since equation (33) is a system of β equations in the β unknowns μ, the existence and, in addition, the uniqueness of the solution are demonstrated if it can be shown that the corresponding homogeneous system

$$\mu = -\eta_\Gamma \bar{W} \tag{34}$$

has only the trivial solution. It will be shown in appendix C that equation (34) has only the trivial solution.

The discussion above applies also to an unbounded region if the function U is required to be regular at infinity and to have the value zero at infinity. The definition of regularity given in the appendix corresponds to the definition of potential theory.

Solution of Integral-Equation Analog for Dirichlet Problem by Iteration

Since the systems of equations (32) and (33) are equivalent, the system of equations (32) will be considered. If M is defined by

$$M = D_\Gamma^{-1} m \tag{35}$$

then equation (32) can be written

$$M = -(\bar{W} - \bar{U}) \tag{36}$$

By equations (20) and (30), if $A = \left(A_{rs}\right)$ and $B = \left(B_{rs}\right)$ then

$$\bar{W} = Am + Bm' \tag{37}$$

where, with r and $s = 1, 2, \ldots \gamma$,

$$\left(A_{rs}\right) = \varphi\left(|p_r - p_s|, |q_r - q_s|\right) \tag{38}$$

and with $r = 1, 2, \ldots \gamma$ and $s = 1, 2, \ldots \nu$,

$$\left(B_{rs}\right) = \varphi\left(|p_r - p_s'|, |q_r - q_s'|\right) \tag{39}$$

Hence, by equation (25)

$$\bar{W} = -CM \tag{40}$$

where

$$C = -\left(AD_T - B\sigma\right) \tag{41}$$

Thus, equation (36) can be written

$$M = \bar{U} + CM \tag{42}$$

If $M^{(k)}$ (where $k = 0, 1, 2, \ldots$) is defined by

$$M^{(0)} = \bar{U} \tag{43}$$

$$M^{(k+1)} = \bar{U} + CM^{(k)} \tag{44}$$

Then $M^{(k)}$ is the result of the kth iteration. It is known that this sequence converges at least geometrically and an estimate of the rate of convergence is being investigated.

If $m^{(k)}$ and $m'^{(k)}$ are defined by

$$m^{(k)} = D_T M^{(k)}$$

$$m'^{(k)} = -\sigma M^{(k)}$$

as in equations (24) and (25), and $W^{(k)}(x,y)$ is defined by

$$W^{(k)}(x,y) = \sum_{r=1}^{\gamma} m_r{}^{(k)} \varphi\left[\left(x - p_r\right),\left(y - q_r\right)\right] + \sum_{r=1}^{\nu} \left(m_r'\right)^{(k)} \varphi\left[\left(x - p_r'\right),\left(y - q_r'\right)\right] \tag{45}$$

then the harmonic extension $U^{(k)}$ of $W^{(k)}$ is

$$U^{(k)}(x,y) = W^{(k)}(x,y) \tag{46}$$

for all inner points (x,y), and $U^{(k)}$ is taken as the kth approximation of $U(x,y)$ on the inner points of G.

A best possible bound for the error of $U^{(k)}$ of the kth iteration can be given in terms of the result for the (k + 1) iteration, $M^{(k+1)}$ and $M^{(k)}$. By equation (40),

$$\bar{W}^{(k)} = -CM^{(k)} \tag{47}$$

and hence the values $\bar{U}^{(k)}$ on the boundary points of the harmonic extension of $\bar{W}^{(k)}$ are defined by the relation

$$M^{(k)} = -\left[\bar{W}^{(k)} - \bar{U}^{(k)}\right] = \bar{U}^{(k)} + CM^{(k)} \tag{48}$$

Comparison of this equation with equation (44) yields

$$\bar{U} - \bar{U}^{(k)} = M^{(k+1)} - M^{(k)} \tag{49}$$

Since $\bar{U} - \bar{U}^{(k)}$ is the difference of two solutions of the Laplace difference equation for G, $\bar{U} - \bar{U}^{(k)}$ also satisfies the Laplace difference equation and by the maximum modulus principle must attain its maximum on the boundary. Thus the element of $\bar{U} - \bar{U}^{(k)}$ which is greatest in absolute value is a best possible bound for the error and this can be computed from equation (49). Thus at each step of the iteration process a bound can be computed for the error of the approximation provided in the preceding step.

Outline of Calculations for Dirichlet Problem

The calculation of the solution of the Dirichlet problem may be divided into three stages. The first stage consists of the calculation of the matrix C defined by equation (41); the second stage concerns the solution, or the approximation of the solution, of the system of equation (42); and the final stage consists of computing the values of the desired function on the inner points in the following way. By equation (35), equation (23) can be written

$$\mu = \eta_T M \tag{50}$$

and by equations (18) and (19)

$$m' = -\sigma M \tag{51}$$

$$m = D_T M \tag{52}$$

Since M is computed in the second stage, these equations give m and m' and by the use of equation (40), a formula for $W(P)$ is obtained. Since at inner points W and U coincide, the values of U on the inner points are computed by this formula. The calculations of the second stage may be carried out in several ways, among them the iteration procedure defined by equations (43) and (44). It should be noted that any estimate of the error based upon equation (49) is equivalent to computing one step of the iteration procedure.

The remainder of this section concerns the calculation of the matrix C. It may be noted that C depends only on the geometry of the given region and not on the values of $\bar{U}$ and, once computed, may be used for any Dirichlet problem for this region.

By equation (41) C is calculated from A, B, σ, and D_Γ. By equations (23) and (24), σ and D_Γ are calculated from η_Γ and η_t as defined by equations (1) and (2). As a preliminary step in the calculation of A and B, two auxiliary matrices, $a = a_{rs}$ and $b = b_{rs}$ defined, with r and $s = 1, 2, \ldots \gamma$, by

$$a_{rs} = \left(|p_r - p_s|, |q_r - q_s|\right) \tag{53}$$

and, with $r = 1, 2, \ldots \gamma$ and $s = 1, 2, \ldots \nu$, by

$$b_{rs} = \left(|p_r - p_s'|, |q_r - q_s'|\right) \tag{54}$$

are calculated. These two matrices are in turn calculated from three other matrices $\alpha_{\gamma\gamma}$, $\beta_{\gamma\nu}$, and $\beta_{\gamma\nu}'$ defined by

$$\alpha_{\gamma\gamma} = \begin{pmatrix} (p_1,q_1)(p_1,q_1) \cdots (p_1,q_1) \\ (p_2,q_2)(p_2,q_2) \cdots (p_2,q_2) \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ (p_\gamma,q_\gamma)(p_\gamma,q_\gamma) \cdots (p_\gamma,q_\gamma) \end{pmatrix} \tag{56}$$

where $\alpha_{\gamma\gamma}$ is a $\gamma \times \gamma$ matrix, each column being a list of the coordinates of the boundary points,

$$\beta_{\gamma\nu} = \begin{pmatrix} (p_1,q_1)(p_1,q_1) & \cdots & (p_1,q_1) \\ (p_2,q_2)(p_2,q_2) & \cdots & (p_2,q_2) \\ \cdots \cdots \cdots & \cdots & \cdots \\ (p_\gamma,q_\gamma)(p_\gamma,q_\gamma) & \cdots & (p_\gamma,q_\gamma) \end{pmatrix} \tag{57}$$

where $\beta_{\gamma\nu}$ is a $\gamma \times \nu$ matrix each column again being a list of the coordinates of the boundary points, and

$$\beta_{\gamma\nu}' = \begin{pmatrix} (p_1',q_1')(p_2',q_2') & \cdots & (p_\nu',q_\nu') \\ (p_i',q_1')(p_2',q_2') & \cdots & (p_\nu',q_\nu') \\ \cdots \cdots \cdots & \cdots & \cdots \\ (p_1',q_1')(p_2',q_2') & \cdots & (p_\nu',q_\nu') \end{pmatrix} \tag{58}$$

where $\beta_{\gamma\nu}'$ is a $\gamma \times \nu$ matrix each row of which is a list of the coordinates of the normal points. By equations (53) and (54)

$$a = \alpha_{\gamma\gamma} - \alpha_{\gamma\gamma t} \tag{59}$$

$$b = \beta_{\gamma\nu} - \beta_{\gamma\nu}' \tag{60}$$

and by equations (38) and (39),

$$A_{rs} = \varphi(a_{rs}) \tag{61}$$

$$B_{rs} = \varphi(b_{rs}) \tag{62}$$

The procedure of stage one may be summarized as follows: For the given region, number the normal inner points from 1 to ν, the boundary points from 1 to γ, and the normal segments from 1 to β. Then from a list of the coordinates of the normal inner and boundary points write the matrices $\alpha_{\gamma\gamma}$ (eq. (56)), $\beta_{\gamma\nu}$ (eq. (57)), and $\beta_{\gamma\nu}'$ (eq. (58)), and calculate a and b. By the use of table I for φ, calculate A and B by equations (61) and (62). The next step is to write η_Γ and η_N by equations (1) and (2) and calculate D_Γ and σ by equations (22) and (24). Alternatively D_Γ can be written by noting that

it is a diagonal matrix such that the rth diagonal element is the number of normal points adjacent to the rth boundary point ($r = 1, 2, \ldots \gamma$), and $\sigma = (\sigma_{ik})$ may also be written directly by the following definition which is equivalent to equation (24):

$$\sigma_{ik} = \begin{cases} 1 & \text{if } (p_k, q_k) \text{ is adjacent to } (p_i', q_i') \\ 0 & \text{if } (p_k, q_k) \text{ is not adjacent to } (p_i', q_i') \end{cases} \tag{63}$$

From A, B, σ, and D_Γ, C is calculated by equation (41). As a partial check on the calculation of C it is proved in the appendix that the sum of the elements in each row of C is zero.

The matrices for the region indicated in figure 1 are given in tables II and III.

The successive approximations of M and the error of the successive approximations of $\bar{U}$ are also given in table IV.

Reduction of Neumann Problem to Integral-Equation Analog

For a finite region the data of the Neumann problem are the set of differences for each normal segment of the value of a function at the inner normal end point minus its value at the boundary end point, and the Neumann problem consists of determining the value of the function, subject to the Laplace difference equation, on the points of the region. For an infinite region, the additional restriction is made that the function be regular at infinity. As in classical potential theory it is a consequence of the linearity of the Laplace difference equation that the solution, if it exists, is determined only up to a constant. For an infinite region, the solution obtained by the method of this paper is the solution whose value at infinity is zero and, for a finite region since a simple-layer potential is used, the solution has the property that the potential representing this function on G is also defined on the complementary region and has the value zero at infinity. It will also be shown in the appendix that, as in classical potential theory, a necessary condition that the Neumann problem have a solution is that the sum of the given differences be zero.

The following definitions will be required. Let

$$(\Delta_1 \bar{U})_t = (\Delta_1 U_1, \Delta_1 U_2, \ldots \Delta_1 U_\gamma) \tag{64}$$

where $\Delta_i U_k$ is the sum of the differences of U for the inner normal segments on the kth boundary point; that is, $\Delta_i U_k$ is the sum of the values of U on the normal points adjacent to the kth boundary point minus the product of the number of normal inner points adjacent to this boundary point and the value of the function on this boundary point. The segments for which at least one end point is a boundary point and which are not inner normal segments of G will be called exterior normal segments of G. Let W be a function whose domain includes the end points of the exterior normal segments and let

$$\left(\Delta_e \bar{W}\right)_t = \left(\Delta_e W_1, \Delta_e W_2, \ldots \Delta_e W_\gamma\right) \tag{65}$$

where $\Delta_e W_k$ is the sum of the values of W on those points which are both end points of exterior normal segments and are adjacent to the kth boundary point minus the product of the number of these adjacent points and the value of W on the kth boundary point. A formula for the calculation of equations (64) and (65) will be given in the next section. It may be remarked that in the notation of equation (64) the condition on the sum of the differences is

$$\sum_{k=1}^{\gamma} \Delta_i U_k = 0 \tag{66}$$

and if W is a simple-layer potential, by equations (15) and (11),

$$\Delta_i W_k + \Delta_e W_k = m_k \tag{67}$$

where $k = 1, 2, \ldots \gamma$ and $\Delta_i W_k$ is calculated in the same way that $\Delta_i U_k$ is calculated. Unless a statement is made to the contrary all simple-layer potentials considered will be subject to the condition

$$\sum_{k=1}^{\gamma} m_k = 0 \tag{68}$$

It will be shown in the appendix that this condition insures that $\lim_{P \to \infty} W(P) = 0$. Since a simple-layer potential satisfies the Laplace difference equation on all inner points, it follows that W(P) is a solution of the Laplace difference equation for the given region. Hence $\Delta_i W$ represents the differences of a solution of the Laplace difference equation on the inner normal segments and by equation (66)

$$\sum_{k=1}^{\gamma} \Delta_1 W_k = 0 \tag{69}$$

By equations (67), (68), and (69),

$$\sum_{k=1}^{\gamma} \Delta_e W_k = 0 \tag{70}$$

For the representation of the solution of the Neumann problem by a simple-layer potential it will be assumed that $\Delta_1\bar{U}$ is given. This can be calculated from the usual form of the data, that is, from the differences prescribed for the inner normal segments. However, it will be shown in the appendix that on the inner points U is determined by $\Delta_1\bar{U}$ only; that is, if two distinct sets of differences are given so that $\Delta_1\bar{U}_1 = \Delta_1\bar{U}_2$, then $U_1 = U_2$ on the inner points up to a constant provided U_1 and U_2 satisfy the Laplace difference equation on the inner points of G.

If the simple-layer potential W represents the solution of a given Neumann problem then

$$\Delta_1\bar{W} = \Delta_1\bar{U} \tag{71}$$

and by equation (67)

$$\Delta_1\bar{U} + \Delta_e\bar{W} = m \tag{72}$$

where $m_t = m_1, m_2, \ldots m_\gamma$ and m_k is the coefficient of $\varphi\left[(x - p_k)(y - q_k)\right]$ in the representation (eq. (15)) of $W(x,y)$.

Conversely, if for given values of $\Delta_1\bar{U}$, there are γ masses $m_1, m_2, \ldots m_\gamma$ so that the simple-layer potential of these masses satisfies equation (72) then by equations (67) and (72)

$$\Delta_1\bar{U} = \Delta_1\bar{W} \tag{73}$$

and hence W computed from these masses by equation (15) represents a solution for the given values of $\Delta_1\bar{U}$. In addition, to show that equation (72) has a unique solution for given values of $\Delta_1 U$ when equation (66) is satisfied, since the number of equations coincides with the number of unknowns, it is only necessary to show that the corresponding

homogeneous system has only the trivial solution. The proof of this assertion will also be found in the appendix.

Solution of Integral Equation Analog of Neumann Problem by Iteration

As a preliminary calculation a formula for equation (72) will be derived. Assume that there are τ exterior normal segments $(s_1', s_2', \ldots s_\tau')$ and let these segments have a fixed arbitrarily assigned orientation. In addition, assume that the end points of the exterior normal segments are denoted by (p_k'', q_k'') where $k = 1, 2, \ldots \xi$. It will also be assumed that the numbering has been carried out so that the boundary points are the first γ points of this set, that is, $(p_k, q_k) = (p_k'', q_k'')$ with $k = 1, 2, \ldots \gamma$. If $A' = (A_{rs}')$ and m and $\bar{W}''$ are defined by

$$A_{rs}' = \varphi\left[(p_r'' - p_s), (q_r'' - q_s)\right] \tag{74}$$

$$m_t = (m_1, m_2, \ldots m_\gamma) \tag{75}$$

$$\bar{W}'' = (W_1'', W_2'', \ldots W_\xi'') \tag{76}$$

where $r = 1, 2, \ldots \xi$, $s = 1, 2, \ldots \gamma$, and $W_k'' = W(p_k'', q_k'')$ where $k = 1, 2, \ldots \xi$, then

$$\bar{W}'' = A'm \tag{77}$$

Let $\eta' = (\eta_{rk}')$ be the incidence matrix of the graph of the exterior normal segments, that is,

$$\eta_{rk}' = \left\{ \begin{array}{l} 1 \quad \text{if } (p_k'', q_k'') \text{ is terminal point of } s_r' \\ -1 \quad \text{if } (p_k'', q_k'') \text{ is initial point of } s_r' \\ 0 \quad \text{if } (p_k'', q_k'') \text{ is not on } s_r' \quad (r = 1, 2, \ldots \tau; \\ \quad k = 1, 2, \ldots \xi) \end{array} \right\} \tag{78}$$

and let $\eta'' = (\eta_{rk}'')$ be the matrix consisting of the first γ columns of η', that is,

$$\eta_{rk}'' = \left\{ \begin{array}{l} 1 \text{ if } (p_k,q_k) \text{ is terminal point of } s_r' \\ -1 \text{ if } (p_k,q_k) \text{ is initial point of } s_r' \\ 0 \text{ if } (p_k,q_k) \text{ is not on } s_r' \quad (r = 1, 2, \ldots \tau; \\ \quad k = 1, 2, \ldots \gamma) \end{array} \right\} \tag{79}$$

It is a consequence of these definitions that

$$\Delta_e \bar{W} = -\eta_t'' \eta' \bar{W}'' \tag{80}$$

and by equation (77)

$$\Delta_e \bar{W} = C'm \tag{81}$$

where

$$C' = -\eta_t'' \eta' A' \tag{82}$$

To calculate $\Delta_i \bar{W}$, let the points of the inner normal segments oriented from the inner normal points to the boundary points be denoted by (p_k''',q_k''') with $k = 1, 2, \ldots \nu + \gamma$ where the numbering is chosen so that $(p_k''',q_k''') = (p_k,q_k)$ with $k = 1, 2, \ldots \gamma$. Let $\eta = (\eta_{rk})$ be the incidence matrix of the inner normal segments where

$$\eta_{rk} = \left\{ \begin{array}{l} 1 \text{ if } (p_k''',q_k''') \text{ is terminal point of } s_r \\ -1 \text{ if } (p_k''',q_k''') \text{ is initial point of } s_r \\ 0 \text{ if } (p_k''',q_k''') \text{ is not on } s_r \quad (r = 1, 2, \ldots \beta; \\ \quad k = 1, 2, \ldots \nu + \gamma) \end{array} \right\} \tag{83}$$

and let η_Γ be the first γ columns of η. Because of the numbering η_Γ is the matrix defined by equation (2). Also, if $A'' = (A_{rk}'')$ is defined by

$$A_{rk}'' = \varphi\left[(p_r''' - p_k)(q_r''' - q_k)\right] \tag{84}$$

where $r = 1, 2, \ldots \nu + \gamma$ and $k = 1, 2, \ldots \gamma$ and $\bar{W}'''$ is defined by

$$\bar{W}''' = \left(W_1''', W_2''', \ldots W_{\nu+\gamma}'''\right) \tag{85}$$

where

$$W_k''' = W\left(p_k''', q_k'''\right) \tag{86}$$

then by equation (15)

$$\bar{W}''' = A''m \tag{87}$$

But by equation (2), equation (83), and the definition of Δ_1,

$$\Delta_1\bar{W} = -\eta_{Tt}\eta\bar{W}''' \tag{88}$$

and hence, by equation (87),

$$\Delta_1\bar{W} = C''m \tag{89}$$

where

$$C'' = -\eta_{Tt}\eta A'' \tag{90}$$

Since equation (67) holds for arbitrary values of m,

$$C' + C'' = I_\gamma \tag{91}$$

where I_γ is the $\gamma \times \gamma$ identity matrix. Thus, C' may be calculated by equation (82) or from equations (90) and (91).

By equation (81), for given values of $\Delta_1\bar{U}$, equation (72) can be written

$$m = \Delta_1\bar{U} + C'm \tag{92}$$

The solution of this equation gives m and consequently $W(x,y)$ which is $U(x,y)$ on the given region.

If this system is to be solved by iteration, let

$$m^{(0)} = \Delta_1 \bar{U} \tag{93}$$

$$m^{(k+1)} = m^{(0)} + C'm^{(k)} \tag{94}$$

This system is known to converge at least geometrically and an estimate of the rate of convergence is being investigated.

Outline of Calculations for Neumann Problem

The calculations by the present method may be summarized as follows: First, C' is calculated; second, the solution m of the system of equations (92) is calculated or approximated; and, third, m is used to calculate W on the region by formula (15). The procedure for the calculation of C' by two methods is given below. The double calculation of C' is a check for the correctness of the calculation.

In the first method A' is calculated in the following way. Let $b' = (b_{rk}')$ and $c' = (c_{rk}')$ be defined by

$$b' = \begin{pmatrix} (p_1'',q_1'')(p_1'',q_1'') & \cdots & (p_1'',q_1'') \\ (p_2'',q_2'')(p_2'',q_2'') & \cdots & (p_2'',q_2'') \\ \cdots \quad \cdots & \cdots & \cdots \\ (p_\xi'',q_\xi'')(p_\xi'',q_\xi'') & \cdots & (p_\xi'',q_\xi'') \end{pmatrix} \tag{95}$$

where b' is the $\xi \times \gamma$ matrix, each column of which is a list of the coordinates of the end points of the exterior normal segments, and

$$c' = \begin{pmatrix} (p_1,q_1)(p_2,q_2) & \cdots & (p_\gamma,q_\gamma) \\ (p_1,q_1)(p_2,q_2) & \cdots & (p_\gamma,q_\gamma) \\ \cdots \quad \cdots & \cdots & \cdots \\ (p_1,q_1)(p_2,q_2) & \cdots & (p_\gamma,q_\gamma) \end{pmatrix} \tag{96}$$

where c' is the $\xi \times \gamma$ matrix, each row of which is a list of the coordinates of the boundary points. Let a' be defined by

$$a' = b' - c' \tag{97}$$

Then by equation (74)

$$A_{rk}' = \varphi(a_{rk}') \tag{98}$$

If η' and η'' are written by definitions (78) and (79), C' is then calculated by equation (82).

In the second method for the calculation of C', let $b'' = (b_{rk}'')$

$$b'' = \begin{pmatrix} (p_1''', q_1''') & (p_1''', q_1''') & \cdots & (p_1''', q_1''') \\ (p_2''', q_2''') & (p_2''', q_2''') & \cdots & (p_2''', q_2''') \\ \cdots & \cdots & \cdots & \cdots \\ (p_{\gamma+\nu}''', q_{\gamma+\nu}''') & (p_{\gamma+\nu}''', q_{\gamma+\nu}''') & \cdots & (p_{\gamma+\nu}''', q_{\gamma+\nu}''') \end{pmatrix} \tag{99}$$

where b'' is the $(\gamma + \nu) \times \gamma$ matrix, each column of which is a list of the coordinates of the end points of the interior normal segments, and

$$c'' = \begin{pmatrix} (p_1, q_1) & (p_2, q_2) & \cdots & (p_\gamma, q_\gamma) \\ (p_1, q_1) & (p_2, q_2) & \cdots & (p_\gamma, q_\gamma) \\ \cdots & \cdots & \cdots & \cdots \\ (p_1, q_1) & (p_2, q_2) & \cdots & (p_\gamma, q_\gamma) \end{pmatrix} \tag{100}$$

where c'' is $(\gamma + \nu) \times \gamma$ matrix, each row of which is a list of the coordinates of the boundary points. If $a'' = (a_{rk}'')$ is defined by

$$a'' = b'' - c'' \tag{101}$$

Then by equation (84)

$$A_{rk}'' = \varphi(a_{rk}'') \tag{102}$$

If η and η_Γ are written by equations (2) and (83) then C'' is calculated by equations (90) and (91), and

$$C' = I_\gamma - C'' \tag{103}$$

A partial check on the calculation of C'' is that the sum of the rows of C'' is zero. This follows from the remark that equation (69) holds for arbitrary values of m, if equation (89) is used and appropriate values of m are chosen. A check on the calculation of C' by the first method is that the sum of the elements of each column of C' is 1. This is a consequence of applying the preceding remark to equation (103). It may be noted that C' depends only on the geometry of the region. Thus, after numbering the inner and exterior normal segments and the end points of these segments, C' is calculated, the solution of the system of equations (92) can be approximated by iteration (eqs. (93) and (94)). The desired function is given by equation (15). To illustrate the method, a Neumann problem for the region of figure 2 has been worked and the details of the calculations are indicated in tables V to VII.

The estimate of the error used is $\left(\Delta_i\bar{U} - \Delta_i\bar{U}^{(k)}\right)$ where $\Delta_i\bar{U}^{(k)}$ refers to the kth approximation $U^{(k)}$. If $m^{(k)}$ is the kth approximation of m, and $U^{(k)} = W^{(k)}$ is computed using equation (15) where $m^{(k)}$ is the kth approximation of U, then by equation (72)

$$\Delta_i\bar{U}^{(k)} = m^{(k)} - \Delta_e\bar{W}^{(k)} \tag{104}$$

But by equations (93) and (94)

$$\Delta_i\bar{U} = m^{(k+1)} - C'm^{(k)} \tag{105}$$

But by equation (81)

$$\Delta_e\bar{W}^{(k)} = C'm^{(k)} \tag{106}$$

Hence

$$\Delta_i\bar{U} - \Delta_i\bar{U}^{(k)} = m^{(k+1)} - m^{(k)} \tag{107}$$

COMMENTS ON WORKED EXAMPLES

The purpose of the worked examples is to indicate the details of the calculations required. It should be noted that the present method should not be used for problems of this size; that is, the method is most useful when the number of inner points is much greater than the number of boundary points.

The rate of convergence for the Dirichlet problem seems to be great enough for practical purposes while the rate of convergence for the Neumann problem would seem to indicate that a modified procedure should be used. A tentative modification of the integral equation analog has increased the rate of convergence for the Neumann problem and is being investigated.

The calculation of the C' matrix was checked by the calculation of C''. This last matrix is not given since its negative differs from the C' matrix by only a constant for the diagonal elements.

Since the values of the fundamental solution were taken to four decimal places, the third decimal place is not exact.

The worked example of the Dirichlet problem is for the region indicated in figure 1, and the boundary values are given in column zero of table III $\left(M^{(0)} = \bar{U}\right)$. The example of the Neumann problem is for the region in figure 3, and the boundary differences are given in table VI. Figures 4 and 5 show the rates of convergence for these examples.

Case Institute of Technology,
Cleveland, Ohio, June 18, 1956.

APPENDIX A

GREEN'S IDENTITIES

First and Second Identities

The proof of Green's first and second identities for a finite set of points given by Courant, Friedrichs, and Lewy (ref. 1) is indicated below as a convenient way to explain the notation. Let G_1 be any set of unit segments and let G, not necessarily a region, be the end points of these segments. Let $U(x,y) = U(P)$ be a function defined on G and let δU be defined for a horizontal segment as the value of U at the right end point minus the value at the left end point and, for a vertical segment, as the value at the upper end point minus the value at the lower end point. If $V(P)$ is another function defined on G, consider the sum

$$\sum_{G} V(P)R\left[U(P)\right]$$

where G below the summation sign means that the sum is to be taken over all points of G and R at a boundary point is to be interpreted according to its definition in the list of symbols. Consider the terms of the sum associated with the horizontal segment whose left end point is P and whose right end point is Q. One term arises from each end point and the terms may be ordered as follows:

$$V(P)\left[U(Q) - U(P)\right] + V(Q)\left[U(P) - U(Q)\right] = -\left[V(P) - V(Q)\right]\left[U(P) - U(Q)\right]$$

By a similar relation for the vertical segments

$$\sum_{G} V(P)R\left[U(P)\right] = -\sum_{G_1} (\delta V)(\delta U) \tag{A1}$$

where the sum on the right is to be taken over all the segments of G_1. This is Green's first identity.

By symmetry U and V can be interchanged on the left to get a similar relation, and the difference of these equations yields Green's second identity

$$\sum_{G} \left[VR(U) - UR(V)\right] = 0 \tag{A2}$$

If G is a region, U satisfies the Laplace difference equation on the inner points of G and $V \equiv 1$; then, equation (A2) becomes

$$\sum_{\Gamma} R(U) = 0$$

which in the Δ notation is equation (66). Hence, the condition expressed by this equation is a necessary condition for the existence of a solution of the Neumann problem.

Also, if G is a region, U satisfies the Laplace difference equation on the inner points and $V = U$; then, equation (A1) becomes in the Δ notation,

$$\bar{U}_t \Delta_1 \bar{U} = -\sum_{G_1} (\delta U)^2$$

If $\bar{U} = 0$ then $\delta U = 0$ for all segments of G_1. Hence, U is a constant and since $U = 0$, $\bar{U} \equiv 0$. If $\Delta_1 \bar{U} = 0$ then U is a constant. It is a consequence of these statements that the solution of the Dirichlet problem is unique if it exists and the solution of the Neumann problem is unique up to a constant if it exists. For finite regions these results are well known. However, if the regularity condition is imposed on the function at infinity these statements can also be asserted for infinite regions. This will be proved in the next section by showing that the above equation holds for infinite regions if U is regular at infinity.

Green's Third Identity

The proof of Green's third identity is given as follows: Let G be a finite region and let $V(P,Q) = L(P,Q)$. Then by equation (A2)

$$\sum_{Q \text{ in } G} U(Q) R_Q[L(P,Q)] = \sum_{Q \text{ in } G} L(P,Q) R[U(Q)]$$

and since G is the sum of I and Γ (see symbol list)

$$\sum_{Q \text{ in } I} U(Q) R_Q[L(P,Q)] = W_1 + W_2 + W_3 \qquad (A3)$$

where

$$W_1(P) = \sum_{Q \text{ in } I} L(P,Q)R[U(Q)] \tag{A4}$$

$$W_2(P) = \sum_{Q \text{ in } \Gamma} L(P,Q)R[U(Q)] \tag{A5}$$

$$W_3(P) = - \sum_{Q \text{ in } \Gamma} U(Q)R_Q[L(P,Q)] \tag{A6}$$

Therefore, $W_1(P)$ may be interpreted as the potential of a mass distribution on I; $W_2(P)$ may be interpreted as the potential of a mass distribution on the boundary Γ; and $W_3(P)$ is the potential of a dipole distribution on the segments of N directed from the boundary to the normal points of dipole density $-U(Q)$. By equation (9) the left side of equation (A3) is zero if P is not in I and $U(P)$ if P is in I. Thus, for any function $U(P)$ defined on a region G

$$W_1(P) + W_2(P) + W_3(P) = \begin{cases} U(P) & \text{if } P \text{ is in } I \\ 0 & \text{if } P \text{ is not in } I \end{cases} \tag{A7}$$

This can be stated as follows: Any function defined on G can be represented as the sum of a potential due to a mass distribution on the inner points, a potential due to a mass distribution on the boundary points, and a dipole distribution on the normal segments.

Let a function $U(P)$ be defined as harmonic on a region G if $R[U(P)] = 0$ for all inner points P of G. For such a function, by equation (A4), $W_1 = 0$ and

$$W_2(P) + W_3(P) = \begin{cases} U(P) & \text{if } P \text{ is in } I \\ 0 & \text{if } P \text{ is not in } I \end{cases} \tag{A8}$$

For finite regions with finite boundaries equation (A8) holds provided that

$$\lim_{r\to\infty} r^2 U = 0 \tag{A9}$$

uniformly with respect to θ, where r and θ are polar coordinates. This is proved by considering a sequence of squares with center at the origin whose sides approach infinity. Consider equation (A8) applied to the part of G in each of these squares. By equations (A9) and (6) it is seen that the parts of W_2 and W_3 computed on the boundary of the square, which is slightly altered so that the region involved is harmonic, goes to zero and the theorem is proved. This result may be strengthened as indicated below.

If $r^2\delta U$ is uniformly bounded and $\lim_{r\to\infty} U = c$ uniformly with respect to θ, it will be said that U is regular at infinity. If the function $\bar{U} = U - c$ is considered then $\lim_{r\to\infty} \bar{U} = 0$ and $\bar{U}$ is regular at infinity. Under these hypotheses it can be shown that W_2 and W_3 on the squares again approach zero and for $\bar{U}$, equation (A8) holds. In terms of equation (A8) this becomes

$$\sum_{Q \text{ in } \Gamma} L(P,Q)R\left[U(Q)\right] - \sum_{Q \text{ in } \Gamma} U(Q)R_Q\left[\overline{L(P,Q)}\right] +$$

$$\sum_{Q \text{ in } \Gamma} cR_Q\left[L(P,Q)\right] = \begin{cases} U(P) - c & \text{if } P \text{ is in } I \\ 0 & \text{if } P \text{ is not in } I \end{cases} \tag{A10}$$

To evaluate the last term on the left consider a sufficiently large square modified as above containing Γ. Let the part of G contained in the square be denoted by G' and let the boundary of G' be denoted by $\Gamma + \Gamma'$. If equation (A8) is applied setting $U = c$,

$$W_C + W_{C'} = \begin{cases} c & \text{if } P \text{ is in interior of } G' \\ 0 & \text{if } P \text{ is not in interior of } G' \end{cases}$$

where

$$W_C = - \sum_{Q \text{ in } \Gamma} cR_Q L(P,Q)$$

$$W_C' = - \sum_{Q \text{ in } \Gamma} cR_Q L(P,Q)$$

But by equation (A8) applied to the square and its interior

$$W_C' = \begin{cases} c & \text{if } P \text{ is in interior of square} \\ 0 & \text{if } P \text{ is not in interior of square} \end{cases}$$

Hence, if P is in both the interior of G' and the interior of the square, $W_C = 0$. Since the square may be taken as large enough to include any finite point of the interior of G, $W_C' = 0$.

Hence, in equation (A10), the last term on the left side is zero and equation (A8) for harmonic functions regular at infinity becomes

$$W_2(P) + W_3(P) + c = \begin{cases} U(P) & \text{if } P \text{ is in } I \\ 0 & \text{if } P \text{ is not in } I \end{cases} \qquad \text{(A11)}$$

This formula can be generalized to the case where U is not regular at infinity. From the discussion above,

$$W(P) = - \sum_{Q \text{ in } \Gamma} R_Q\left[L(P,Q)\right] = \begin{cases} 1 & \text{if } P \text{ is in } I \\ 0 & \text{if } P \text{ is not in } I \end{cases} \qquad \text{(A12)}$$

This suggests, as in potential theory, defining the mass of U as

$$M(U) = - \sum_{Q \text{ in } \Gamma} R\left[U(Q)\right] \qquad \text{(A13)}$$

If U can be represented in the form

$$U(P) = ML(P,P') + U_1(P)$$

where $U_1(P)$ is regular at infinity then equation (A11) can be shown to hold. It may be noted that definition (A13) resembles the definition of flux in potential theory.

If the components of $\bar{U}$ are identical then U is a constant function and $W_1 = W_2 = 0$ for all values of P. Also, $W_3(P)$ can be interpreted as the potential due to a dipole distribution for which all the components of μ equal the same constant, say c. Then by equation (A8) such a potential is c on all inner points and zero on all other points. Thus, by equation (40) the sum of the columns of C must be zero.

A similar analysis of the Green's identities for infinite regions applied to simple-layer potentials shows that condition (68) guarantees the validity of Green's identities for functions represented by simple-layer potentials.

APPENDIX B

HOMOGENEOUS INTEGRAL-EQUATION ANALOG FOR NEUMANN PROBLEM

The homogeneous system corresponding to equation (72) is obtained by setting $\Delta_i U = 0$ in equation (72); that is,

$$\Delta_e \bar{W} = m \tag{B1}$$

By equation (68) and the discussion of appendix A W is regular at infinity and its value at infinity is zero. Thus, for finite and infinite values of G, Green's identity gives

$$\sum_{G_1} (\delta W)^2 = -\bar{W}_t \Delta_1 \bar{W} \tag{B2}$$

But by equations (67) and (B1)

$$\Delta_i \bar{W} = 0$$

Hence, by equation (B2) $\delta W = 0$ and W is a constant on G. Now on the set consisting of boundary points of G or the points of the complement of G, W is harmonic. Since the set of boundary points of the above set is also the set of boundary points Γ of G and since W is constant on P, it follows that W is constant on the complement of G. But the point at infinity is in one of these regions and W is zero at this point. Hence, W is identically zero. Hence, $\Delta_e \bar{W} = 0$ and $m = 0$. Thus the homogeneous integral analog system for the Neumann problem has only the trivial solution and the existence of a unique solution for the system of equation (72) is proved.

APPENDIX C

HOMOGENEOUS INTEGRAL-EQUATION ANALOG FOR DIRICHLET PROBLEM

The homogeneous integral-equation analog for the Dirichlet problem is given as follows: In order to prove the existence and uniqueness of the system of equations (33) or (36) it suffices to prove that the system of equations

$$M = -W \tag{C1}$$

obtained by setting $\bar{U} = 0$ has only the trivial solution.

As a preliminary step a representation of the harmonic extension of W will be derived. Let V be that solution of the Laplace difference equation on G for which the boundary values $\bar{V}$ of V are

$$\bar{V} = M \tag{C2}$$

By equation (A8)

$$\sum_{Q \text{ in } \Gamma} L(P,Q)R\left[V(Q)\right] - \sum_{Q \text{ in } \Gamma} V(Q)R_Q\left[L(P,Q)\right] = \begin{Bmatrix} V(P) & \text{for } P \text{ in } I \\ 0 & \text{for } P \text{ in } \Gamma \end{Bmatrix} \tag{C3}$$

It is a consequence of equations (C2) and (20) that

$$W(P) = -\sum_{Q \text{ in } \Gamma} V(Q)R_Q\left[L(P,Q)\right] \tag{C4}$$

If $H(P)$ is defined by

$$H(P) = \sum_{Q \text{ in } \Gamma} L(P,Q)R\left[V(Q)\right] \tag{C5}$$

then $H(P)$ is harmonic in G, and

$$W(P) = \begin{Bmatrix} V(P) - H(P) & \text{for } P \text{ in } I \\ \left[V(P) - H(P)\right] - V(P) & \text{for } P \text{ in } \Gamma \end{Bmatrix} \tag{C6}$$

Also, since $\sum_{Q \text{ in } \Gamma} R\left[V(Q)\right] = 0$, $H(P)$ is regular at infinity and is zero at infinity.

From equations (36) and (C2)

$$W(P) = \begin{Bmatrix} U(P) \text{ for } P \text{ in } I \\ U(P) - V(P) \text{ for } P \text{ in } \Gamma \end{Bmatrix} \tag{C7}$$

Comparison with equation (C6) yields

$$U(P) = V(P) - H(P) \tag{C8}$$

for all points P in G.

If $\bar{U} = 0$ then by equation (C8)

$$V(P) = \sum_{Q_k \text{ in } \Gamma} m(Q_k) L(P, Q_k) \tag{C9}$$

where

$$m(Q_k) = R\left[V(Q_k)\right] = \Delta_i V_k \tag{C10}$$

But

$$m(Q_k) = \Delta_i V_k + \Delta_e V_k \tag{C11}$$

Hence

$$\Delta_e \bar{V} = 0$$

By reasoning similar to that of the preceding section V is identically zero and hence M is zero. This completes the proof of the existence and uniqueness of the solution of the integral-equation analog for the Dirichlet problem.

APPENDIX D

CLASSICAL POTENTIAL THEORY

Since the theory in this paper is an analog of classical potential theory, a brief sketch of this subject and the reduction of the Neumann and Dirichlet problems to integral equations is given here. All regions mentioned below are assumed to be at least as regular as is required for the following statements to hold.

If r is the distance from a given point to a unit positive electrical point charge, the value of the potential of the charge at the given point is $1/r$ in suitable units. This potential function is a solution of the Laplace partial-differential equation

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = 0$$

in any region not containing the charge. If S is a surface and if a charge density $m_1(x,y,z)$ is defined on the surface, then the potential H_1 of this charge distribution is

$$H_1(x,y,z) = \iint_S \frac{m_1(\xi,\eta,\zeta)}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}}\, dA$$

The potential H_1 is also a solution of the Laplace differential equation in any regular region not containing S and is called a simple-layer potential. This can be verified by direct substitution if the derivatives are calculated by inverting the order of differentiation and integration. It is a central theorem of classical potential theory that any solution of the Laplace equation in a region can be represented as a simple-layer potential.

If two numerically equal charges of opposite sign are brought together along a fixed line and the magnitudes of the charges are varied so that the product of the numerical value of the charges and the distance between them is held fixed, then the limit of the potential obtained by letting the distance between the charges go to zero is called the potential of a dipole oriented along the given line and has the form $\partial(1/r)/\partial n$, where $\partial/\partial n$ denotes differentiation along the given line. This element is called a dipole and is also a solution of the Laplace differential equation. As above, if a dipole density m_2 is defined on S, the potential

$$H_2(x,y,z) = \iint_S m_2(\xi,\eta,\zeta)\,\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dA$$

where

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$$

and $\partial/\partial n$ denotes differentiation along the normal to S, is also a solution of Laplace's differential equation in any regular region not containing S. It can be shown that any solution of the Laplace equation can be represented, up to an additive constant, by H_2, that is, by a dipole or double-layer potential.

The Dirichlet or first boundary-value problem of potential theory consists of finding a solution of Laplace's equation in a region which assumes given values on the boundary of the region. The second problem is to find a solution in the region in which the normal derivative on the boundary is a given function. The first problem can be reduced to the problem of determining a dipole density on the boundary whose potential is the desired function; the second problem can be formulated as the problem of determining a charge density on the boundary whose potential is the desired function. In either case an integral equation for the dipole or charge density can be derived by using either the discontinuity of a dipole distribution on the boundary or the discontinuity of the normal derivative of a simple-layer potential on the boundary.

For the Laplace equation in two dimensions the physical interpretation is not so straightforward, but by analogy with the three-dimensional case a "charge" is considered whose potential is $\log_e \frac{1}{r}$ where r is the distance between the charge and the point at which the potential is being evaluated. Dipoles are as defined previously, but instead of considering distributions on surfaces, distributions on curves are considered. All the other statements for the three-dimensional case hold for the two-dimensional case. Thus the potentials of a charge and the dipole distributions, respectively, are

$$H_1(x,y) = \int_C m_1(\xi,\eta)\log_e \frac{1}{r}\,ds$$

$$H_2(x,y) = \int_C m_2(\xi,\eta)\frac{\partial}{\partial n}\left(\log_e \frac{1}{r}\right)ds$$

where $r = \sqrt{(x - \xi)^2 + (y - \eta)^2}$, $\partial/\partial n$ indicates differentiation along the normal to the curve C, and ds is the element of arc length on C. If the limit from the interior of a region is denoted by the subscript i and the limit from the exterior by the subscript e, the discontinuity of a double-layer potential on the boundary of a given region can be expressed by

$$H_{2i}(x,y) = H_2(x,y) + \pi m_2(x,y)$$

$$H_{2e}(x,y) = H_2(x,y) - \pi m_2(x,y)$$

where (x,y) is a point on the boundary. For a simple-layer potential if, at a fixed boundary point (x_o,y_o), differentiation in the direction of this normal is denoted by $\partial/\partial n$ then

$$\left[\frac{\partial H_1(x_o,y_o)}{\partial n}\right]_i = -\pi m_1(x_o,y_o) + \int_C m_1(\xi,\eta)\frac{\partial}{\partial n}\left(\log_e \frac{1}{r}\right) ds$$

$$\left[\frac{\partial H_1(x_o,y_o)}{\partial n}\right]_e = \pi m_1(x_o,y_o) + \int_C m_1(\xi,\eta)\frac{\partial}{\partial n}\left(\log_e \frac{1}{r}\right) ds$$

where C is the boundary of the given region.

In order to solve the Dirichlet problem by representing the solution as a double-layer potential, it is noted that $H_{2i}(x,y)$ must coincide with the given boundary value of the desired function. Substituting for H_2 its representation in terms of m_2 yields

$$\int_C m_2(\xi,\eta)\frac{\partial}{\partial n}\left(\log_e \frac{1}{r}\right) ds + \pi m_2(x,y) = F(s)$$

where $F(s)$ is the given value of the desired function at the point on the boundary curve C, $x = x(s)$, and $y = y(s)$. If the substitutions are carried out, a linear integral equation for m_2 is obtained. The exterior problem leads to a similar equation if the second discontinuity condition for a double-layer potential is used.

For the inner Neumann problem it is noted that $\dfrac{\partial H_1(x_o,y_o)}{\partial n}$ is given if the solution is regarded as represented by a simple-layer potential

and, hence, the first discontinuity condition for the normal derivative of a simple-layer potential becomes an integral equation for the "charge density." Sufficient conditions which ensure the existence and uniqueness of the solutions of these problems as well as the proofs of the above statements can be found in reference 6. The outer problem can be solved by using the second condition.

The potential theoretical approach to the Neumann and Dirichlet problems may be summarized in the following way. The given problem consists of finding a function which satisfies the Laplace equation on a region and, in the case of the Dirichlet problem, assumes given values on the boundary of the region or, in the case of the Neumann problem, has a normal derivative on the boundary which assumes prescribed values on the boundary. In either case the unknown function is regarded as the potential of a simple- or double-layer charge density on the boundary which satisfies a certain integral equation. Thus the original problem is reduced to the problem of solving an integral equation for the charge density. Once the charge density is known, the potential, which is the desired function, can be computed directly.

The methods given in this paper for handling the corresponding problems for the Laplace difference equation are analogs of the above methods.

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TABLE I

LAPLACE SOLUTION, $\varphi(p,q)$

q \ p	0	1	2	3	4	5	6	7	8	9
0	000,000	250,000	363,380	430,281	476,994	512,902	542,116	566,760	588,082	606,874
1		318,310	386,620	440,376	482,395	516,250	544,400	568,422	589,347	607,870
2			424,413	462,207	495,962	525,303	550,811	573,181	593,013	610,778
3				488,075	513,944	538,190	560,359	580,480	598,746	615,387
4					533,548	553,152	571,956	589,633	606,103	621,403
5						568,916	584,679	599,992	614,631	628,505
6							597,853	611,027	623,923	636,386
7								622,338	633,650	644,779
8									643,559	653,468
9										662,283

q \ p	10	11	12	13	14	15	16	17	18	19
0	623,676	638,869	652,735	665,488	677,294	688,283	698,562	708,217	717,319	725,928
1	624,480	639,532	653,292	665,962	677,702	688,639	698,874	708,493	717,566	726,150
2	626,842	641,488	654,938	667,366	678,914	689,695	699,803	709,317	718,300	726,809
3	630,622	644,640	657,605	669,651	680,892	691,424	701,327	710,669	719,509	727,896
4	635,618	648,847	661,190	672,740	683,579	693,782	703,412	712,525	721,171	729,393
5	641,599	653,939	665,569	676,540	686,904	696,713	706,013	714,848	723,257	731,276
6	648,335	659,742	670,607	680,946	690,784	700,152	709,079	717,597	725,734	733,519
7	655,613	666,088	676,171	685,852	695,135	704,031	712,555	720,726	728,565	736,089
8	663,251	672,824	682,136	691,157	699,873	708,281	716,384	724,189	731,709	738,954
9	671,098	679,822	688,393	696,767	704,920	712,836	720,509	727,938	735,127	742,080
10	679,036	686,974	694,845	702,599	710,203	717,634	724,879	731,928	738,780	745,434
11		694,194	701,413	708,581	715,659	722,620	729,443	736,116	742,630	748,983
12			708,033	714,654	721,233	727,742	734,157	740,461	746,643	752,695
13				720,766	726,878	732,959	738,982	744,929	750,786	756,542
14					732,555	738,232	743,883	749,488	755,030	760,497
15						743,531	748,831	754,109	759,348	764,536
16							753,799	758,768	763,719	768,638
17								763,445	768,122	772,784
18									772,540	776,957
19										781,143

TABLE I.- Continued

LAPLACE SOLUTION, $\varphi(p,q)$

q \ p	20	21	22	23	24	25	26	27	28
0	734,096	741,864	749,271	756,348	763,123	769,622	775,866	781,874	787,663
1	734,295	742,045	749,435	756,498	763,262	769,750	775,984	781,983	787,765
2	734,890	742,585	749,928	756,949	763,675	770,131	776,336	782,310	788,069
3	735,873	743,477	750,741	757,694	764,360	770,763	776,921	782,852	788,573
4	737,228	744,709	751,866	758,725	765,309	771,638	777,751	783,605	789,274
5	738,956	746,265	753,290	760,032	766,512	772,750	778,762	784,562	790,166
6	740,973	748,127	754,995	761,600	767,959	774,088	780,003	785,717	791,242
7	743,318	750,271	756,964	763,414	769,634	775,641	781,445	787,060	792,496
8	745,939	752,675	759,177	765,456	771,525	777,395	783,077	788,582	793,918
9	748,807	755,314	761,612	767,709	773,614	779,338	784,887	790,272	795,499
10	751,894	758,163	764,247	770,153	775,886	781,453	786,862	792,119	797,230
11	755,171	761,197	767,061	772,769	778,323	783,728	788,989	794,112	799,101
12	758,611	764,391	770,033	775,538	780,909	786,147	791,255	796,238	801,100
13	762,189	767,723	773,142	778,443	783,627	788,695	793,647	798,487	803,217
14	765,880	771,171	776,368	781,465	786,462	791,357	796,152	800,846	805,441
15	769,661	774,715	779,692	784,587	789,398	794,121	798,757	803,304	807,763
16	773,514	778,336	783,098	787,794	792,420	796,973	801,450	805,850	810,172
17	777,420	782,018	786,571	791,072	795,516	799,900	804,219	808,473	812,659
18	781,362	785,744	790,095	794,406	798,673	802,891	807,055	811,164	815,214
19	785,328	789,503	793,658	797,785	801,880	805,935	809,947	813,912	817,829
20	789,304	793,281	797,248	801,198	805,125	809,022	812,885	816,710	820,494
21		797,068	800,855	804,635	808,399	812,143	815,861	819,549	823,204
22			804,471	808,086	811,694	815,290	818,867	822,421	825,949
23				811,544	815,003	818,455	821,895	825,320	828,725
24					818,317	821,631	824,940	828,239	831,523
25						824,813	827,995	831,172	834,340
26							831,054	834,114	837,169
27								837,060	840,006
28									842,848

q \ p	29	30	31	32	33	34	35	36	37
0	793,249	798,646	803,866	808,919	813,818	818,570	823,184	827,668	832,029
1	793,344	798,735	803,949	808,997	813,891	818,639	823,249	827,729	832,087
2	793,628	798,999	804,197	809,230	814,110	818,843	823,444	827,913	832,262
3	794,098	799,439	804,609	809,617	814,474	819,188	823,767	828,219	832,551
4	794,752	800,051	805,182	810,155	814,980	819,665	824,218	828,645	832,955
5	795,584	800,830	805,912	810,842	815,626	820,274	824,793	829,190	833,471
6	796,590	801,772	806,796	811,673	816,409	821,013	825,491	829,851	834,097
7	797,763	802,871	807,829	812,644	817,324	821,877	826,308	830,624	834,830
8	799,095	804,121	809,003	813,750	818,368	822,863	827,241	831,508	835,669
9	800,577	805,513	810,314	814,985	819,534	823,965	828,284	832,497	836,608
10	802,202	807,041	811,753	816,343	820,817	825,179	829,435	833,589	837,645
11	803,961	808,697	813,314	817,817	822,211	826,500	830,687	834,778	838,775
12	805,842	810,471	814,989	819,401	823,711	827,921	832,036	836,060	839,995
13	807,838	812,356	816,771	821,088	825,309	829,438	833,478	837,431	841,300
14	809,939	814,342	818,651	822,870	827,000	831,044	835,005	838,885	842,686
15	812,135	816,421	820,623	824,741	828,777	832,734	836,614	840,417	844,148
16	814,418	818,585	822,677	826,693	830,634	834,502	838,298	842,024	845,681
17	816,777	820,826	824,807	828,719	832,563	836,341	840,052	843,699	847,282
18	819,205	823,136	827,005	830,813	834,560	838,246	841,872	845,438	848,945
19	821,694	825,506	829,265	832,969	836,618	840,212	843,751	847,235	850,666
20	824,235	827,931	831,579	835,179	838,730	842,232	845,685	849,087	852,440
21	826,822	830,402	833,941	837,438	840,892	844,302	847,668	850,988	854,264
22	829,448	832,914	836,345	839,741	843,098	846,417	849,696	852,935	856,133
23	832,106	835,460	838,786	842,081	845,343	848,572	851,765	854,922	858,043
24	834,790	838,036	841,258	844,454	847,622	850,761	853,869	856,946	859,990
25	837,495	840,634	843,755	846,854	849,930	852,981	856,006	859,002	861,970
26	840,216	843,252	846,274	849,278	852,264	855,228	858,170	861,087	863,980
27	842,949	845,884	848,809	851,721	854,618	857,498	860,358	863,198	866,016
28	845,689	848,526	851,358	854,180	856,990	859,786	862,567	865,331	868,073
29	848,432	851,173	853,915	856,650	859,376	862,091	864,794	867,482	870,155
30		853,827	856,479	859,128	861,772	864,408	867,034	869,649	872,251
31			859,045	861,612	864,175	866,734	869,286	871,830	874,363
32				864,098	866,584	869,068	871,547	874,020	876,486
33					868,995	871,406	873,815	876,219	878,618
34						873,746	876,086	878,424	880,758
35							878,359	880,632	882,903
36								882,842	885,052
37									887,203

TABLE I.- Concluded

LAPLACE SOLUTION, $\Phi(p,q)$

q \ p	38	39	40	41	42	43	44
0	836,274	840,409	844,458	848,369	852,204	855,950	859,609
1	836,389	840,461	844,488	848,416	852,250	855,995	859,650
2	836,494	840,618	844,637	848,558	852,385	856,122	859,773
3	836,769	840,879	844,885	848,794	852,610	856,337	859,978
4	837,152	841,242	845,251	849,125	852,924	856,636	860,264
5	837,641	841,707	845,673	849,549	853,325	857,019	860,632
6	838,256	842,272	846,211	850,057	853,813	857,485	861,076
7	838,932	842,934	846,841	850,657	854,387	858,033	861,600
8	839,728	843,691	847,562	851,345	855,043	858,660	862,199
9	840,621	844,541	848,372	852,117	855,780	859,364	862,873
10	841,607	845,480	849,267	852,971	856,596	860,144	863,620
11	842,683	846,505	850,244	853,904	857,488	860,998	864,437
12	843,845	847,613	851,302	854,914	858,454	861,922	865,322
13	845,089	848,799	852,435	855,998	859,490	862,914	866,273
14	846,411	850,062	853,641	857,151	860,594	863,973	867,288
15	847,806	851,395	854,916	858,372	861,764	865,094	868,364
16	849,271	852,796	856,257	859,656	862,995	866,275	869,498
17	850,802	854,261	857,660	861,001	864,285	867,513	870,687
18	852,394	855,786	859,122	862,403	865,631	868,806	871,930
19	854,042	857,366	860,638	863,859	867,028	870,150	873,222
20	855,744	858,999	862,206	865,365	868,477	871,543	874,563
21	857,495	860,680	863,821	866,918	869,972	872,981	875,948
22	859,290	862,406	865,482	868,515	871,508	874,461	877,375
23	861,126	864,172	867,181	870,152	873,086	875,982	878,841
24	863,000	865,977	868,919	871,827	874,701	877,540	880,344
25	864,908	867,815	870,691	873,537	876,350	879,132	881,882
26	866,846	869,684	872,493	875,278	878,031	880,756	883,451
27	868,811	871,581	874,327	877,047	879,742	882,409	885,050
28	870,800	873,503	876,185	878,843	881,478	884,089	886,676
29	872,810	875,447	878,069	880,665	883,239	885,794	888,327
30	874,839	877,411	879,966	882,503	885,021	887,520	890,000
31	876,883	879,391	881,884	884,362	886,825	889,267	891,695
32	878,941	881,386	883,818	886,237	888,642	891,032	893,405
33	881,010	883,395	885,765	888,127	890,476	892,812	895,134
34	883,087	885,410	887,724	890,029	892,323	894,606	896,877
35	885,171	887,435	889,692	891,941	894,183	896,413	898,634
36	887,261	889,466	891,667	893,862	896,050	898,230	900,401
37	889,353	891,502	893,648	895,789	897,926	900,056	902,179
38	891,447	893,540	895,633	897,725	899,808	901,889	903,964
39		895,581	897,621	899,660	901,696	903,729	905,757
40			899,610	901,599	903,587	905,573	907,555
41				903,540	905,480	907,420	909,357
42					907,375	909,269	911,165
43						911,119	912,970
44							914,778

q \ p	45	46	47	48	49	50	
0	863,186	866,684	870,107	873,458	876,740	879,956	
1	863,225	866,722	870,145	873,495	876,773	879,988	
2	863,343	866,835	870,251	873,597	876,875	880,085	
3	863,539	867,022	870,431	873,769	877,038	880,282	
4	863,813	867,284	870,682	874,009	877,269	880,563	
5	864,165	867,620	871,004	874,318	877,565	880,748	
6	864,589	868,028	871,395	874,695	877,925	881,094	
7	865,090	868,507	871,854	875,134	878,349	881,501	
8	865,654	869,057	872,382	875,640	878,835	881,969	
9	866,309	869,676	872,975	876,210	879,382	882,495	
10	867,024	870,361	873,633	876,841	879,989	883,079	
11	867,807	871,112	874,354	877,534	880,655	883,719	
12	868,656	871,927	875,136	878,285	881,378	884,415	
13	869,569	872,803	875,977	879,094	882,156	885,164	
14	870,542	873,738	876,876	879,959	882,988	885,965	
15	871,575	874,730	877,830	880,877	883,872	886,817	
16	872,665	875,778	878,838	881,847	884,807	887,718	
17	873,808	876,877	879,897	882,867	885,789	888,665	
18	875,005	878,028	881,004	883,934	886,818	889,658	
19	876,247	879,226	882,159	885,047	887,898	890,694	
20	877,538	880,469	883,357	886,205	889,008	891,772	
21	878,872	881,756	884,599	887,401	890,169	892,889	
22	880,248	883,083	885,880	888,639	891,360	894,043	
23	881,665	884,449	887,199	889,913	892,592	895,236	
24	883,115	885,851	888,554	891,223	893,859	896,462	
25	884,600	887,287	889,942	892,565	895,158	897,720	
26	886,117	888,754	891,361	893,939	896,488	899,008	
27	887,664	890,251	892,810	895,342	897,847	900,325	
28	889,238	891,775	894,286	896,773	899,234	901,669	
29	890,837	893,324	895,788	898,228	900,646	903,039	
30	892,458	894,896	897,313	899,708	902,081	904,432	
31	894,101	896,490	898,859	901,209	903,536	905,847	
32	895,765	898,103	900,425	902,730	905,016	907,283	
33	897,441	899,733	902,009	904,269	906,512	908,738	
34	899,125	901,380	903,610	905,825	908,025	910,210	
35	900,843	903,040	905,229	907,397	909,554	911,698	
36	902,565	904,714	906,854	908,982	911,097	913,200	
37	904,295	906,398	908,494	910,579	912,655	914,716	
38	906,032	908,093	910,145	912,188	914,221	916,244	
39	907,779	909,796	911,805	913,806	915,799	917,785	
40	909,533	911,506	913,473	915,433	917,386	919,331	
41	911,291	913,222	915,147	917,067	918,981	920,886	
42	913,054	914,942	916,827	918,708	920,585	922,452	
43	914,819	916,667	918,512	920,353	922,190	924,025	
44	916,587	918,394	920,200	922,003	923,803	925,599	
45	918,355	920,125	921,890	923,656	925,419	927,180	
46		921,855	923,582	925,311	927,039	928,764	
47			925,275	926,968	928,660	930,351	
48				928,626	930,284	931,941	
49					931,908	933,531	
50						935,125	

TABLE II.- MATRICES FOR DIRICHLET PROBLEM

(a) η_N matrix

Inner normal segments	Normal points									
	1	2	3	4	5	6	7	8	9	10
1	-1									
2		-1								
3			-1							
4				-1						
5				-1						
6					-1					
7						-1				
8						-1				
9							-1			
10								-1		
11									-1	
12									-1	
13										-1
14	-1									

(b) η matrix

Inner normal segments	Normal points													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1													
2		1												
3			1											
4				1										
5					1									
6						1								
7							1							
8								1						
9									1					
10										1				
11											1			
12												1		
13													1	
14														1

TABLE II.- MATRICES FOR DIRICHLET PROBLEM - Continued

(c) a matrix

Row	Column													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0,0	1,0	2,0	3,0	4,1	4,2	4,3	3,4	2,4	1,4	0,4	1,3	1,2	1,1
2	1,0	0,0	1,0	2,0	3,1	3,2	3,3	2,4	1,4	0,4	1,4	2,3	2,2	2,1
3	2,0	1,0	0,0	1,0	2,1	2,2	2,3	1,4	0,4	1,4	2,4	3,3	3,2	3,1
4	3,0	2,0	1,0	0,0	1,1	1,2	1,3	0,4	1,4	2,4	3,4	4,3	4,2	4,1
5	4,1	3,1	2,1	1,1	0,0	0,1	0,2	1,3	2,3	3,3	4,3	5,2	5,1	5,0
6	4,2	3,2	2,2	1,2	0,1	0,0	0,1	1,2	2,2	3,2	4,2	5,1	5,0	5,1
7	4,3	3,3	2,3	1,3	0,2	0,1	0,0	1,1	2,1	3,1	4,1	5,0	5,1	5,2
8	3,4	2,4	1,4	0,4	1,3	1,2	1,1	0,0	1,0	2,0	3,0	4,1	4,2	4,3
9	2,4	1,4	0,4	1,4	2,3	2,2	2,1	1,0	0,0	1,0	2,0	3,1	3,2	3,3
10	1,4	0,4	1,4	2,4	3,3	3,2	3,1	2,0	1,0	0,0	1,0	2,1	2,2	2,3
11	0,4	1,4	2,4	3,4	4,3	4,2	4,1	3,0	2,0	1,0	0,0	1,1	1,2	1,3
12	1,3	2,3	3,3	4,3	5,2	5,1	5,0	4,1	3,1	2,1	1,1	0,0	0,1	0,2
13	1,1	2,2	3,2	4,2	5,1	5,0	5,1	4,2	3,2	2,2	1,2	0,1	0,0	0,1
14	1,1	2,1	3,1	4,1	5,0	5,1	5,2	4,3	3,3	2,3	1,3	0,2	0,1	0,0

(d) b matrix

Row	Column									
	1	2	3	4	5	6	7	8	9	10
1	0,1	1,1	2,1	3,1	3,2	3,3	2,3	1,3	0,3	0,2
2	1,1	0,1	1,1	2,1	2,2	2,3	1,3	0,3	1,3	1,2
3	2,1	1,1	0,1	1,1	1,2	1,3	0,3	1,3	2,3	2,2
4	3,1	2,1	1,1	0,1	0,2	0,3	1,3	2,3	3,3	3,2
5	4,0	3,0	2,0	1,0	1,1	1,2	2,2	3,2	4,2	4,1
6	4,1	3,1	2,1	1,1	1,0	1,1	2,1	3,1	4,1	4,0
7	4,2	3,2	2,2	1,2	1,1	1,0	2,0	3,0	4,0	4,1
8	3,3	2,3	1,3	0,3	0,2	0,1	1,1	2,1	3,1	3,2
9	2,3	1,3	0,3	1,3	1,2	1,1	0,1	1,1	2,1	2,2
10	1,3	0,3	1,3	2,3	2,2	2,1	1,1	0,1	1,1	1,2
11	0,3	1,3	2,3	3,3	3,2	3,1	2,1	1,1	0,1	0,2
12	1,2	2,2	3,2	4,2	4,1	4,0	3,0	2,0	1,0	1,1
13	1,1	2,1	3,1	4,1	4,0	4,1	3,1	2,1	1,1	1,0
14	1,0	2,0	3,0	4,0	4,1	4,2	3,2	2,2	1,2	1,2

TABLE II.- MATRICES FOR DIRICHLET PROBLEM - Continued

(e) A matrix

Row	Column													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	0.2500	0.3634	0.4303	0.4824	0.4960	0.5140	0.5140	0.4960	0.4824	0.4770	0.4404	0.3866	0.3183
2	.2500	0	.2500	.3634	.4404	.4622	.4881	.4960	.4824	.4770	.4824	.4622	.4244	.3866
3	.3634	.2500	0	.2500	.3866	.4244	.4622	.4824	.4770	.4824	.4960	.4881	.4622	.4404
4	.4303	.2634	.2500	0	.3183	.3866	.4404	.4770	.4824	.4960	.5140	.5140	.4960	.4824
5	.4824	.4404	.3866	.3183	0	.2500	.3634	.4404	.4622	.4881	.5140	.5253	.5163	.5129
6	.4960	.4622	.4244	.3866	.2500	0	.2500	.3866	.4244	.4622	.4960	.5163	.5129	.5163
7	.4150	.4881	.4622	.4404	.3634	.2500	0	.3183	.3866	.4404	.4824	.5129	.5163	.5253
8	.5140	.4960	.4824	.4770	.4404	.3866	.3183	0	.2500	.3634	.4303	.4824	.4960	.5140
9	.4960	.4824	.4770	.4824	.4622	.4244	.3866	.2500	0	.2500	.3634	.4404	.4622	.4881
10	.4824	.4770	.4824	.4960	.4881	.4622	.4404	.3634	.2500	0	.2500	.3866	.4244	.4622
11	.4770	.4824	.4960	.5140	.5140	.4960	.4824	.4303	.3634	.2500	0	.3183	.3866	.4404
12	.4404	.4622	.4881	.5140	.5253	.5163	.5129	.4824	.4404	.3866	.3183	0	.2500	.3634
13	.3866	.4244	.4622	.4960	.5163	.5129	.5163	.4960	.4622	.4244	.3866	.2500	0	.2500
14	.3183	.3866	.4404	.4824	.5129	.5163	.5253	.5140	.4881	.4622	.4404	.3634	.2500	0

(f) B matrix

Row	Column									
	1	2	3	4	5	6	7	8	9	10
1	0.2500	0.3183	0.3866	0.4404	0.4622	0.4881	0.4622	0.4404	0.4303	0.3634
2	.3183	.2500	.3183	.3866	.4244	.4622	.4404	.4303	.4404	.3866
3	.3866	.3183	.2500	.3183	.3866	.4404	.4303	.4404	.4622	.4244
4	.4404	.3866	.3183	.2500	.3634	.4303	.4404	.4622	.4881	.4622
5	.4770	.4303	.3634	.2500	.3183	.3866	.4244	.4622	.4960	.4824
6	.4824	.4404	.3866	.3183	.2500	.3183	.3866	.4404	.4824	.4770
7	.4960	.4622	.4244	.3866	.3183	.2500	.3634	.4303	.4770	.4824
8	.4881	.4622	.4404	.4303	.3634	.2500	.3183	.3866	.4404	.4622
9	.4622	.4404	.4303	.4404	.3866	.3183	.2500	.3183	.3866	.4244
10	.4404	.4303	.4404	.4622	.4244	.3866	.3183	.2500	.3183	.3866
11	.4303	.4404	.4622	.4881	.4622	.4404	.3866	.3183	.2500	.3634
12	.3866	.4244	.4622	.4960	.4824	.4770	.4303	.3634	.2500	.3183
13	.3183	.3866	.4404	.4824	.4770	.4824	.4404	.3866	.3183	.2500
14	.2500	.3634	.4303	.4770	.4824	.4960	.4622	.4244	.3866	.3183

TABLE II.- MATRICES FOR DIRICHLET PROBLEM - Concluded

(g) C matrix

Row	Column													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0.2500	0.0683	0.0232	0.0101	-0.0420	-0.0338	-0.0259	-0.0259	-0.0338	-0.0420	-0.0467	-0.0101	-0.0232	-0.0683
2	.0683	.2500	.0683	.0232	-.0538	-.0378	-.0259	-.0338	-.0420	-.0467	-.0420	-.0218	-.0378	-.0683
3	.0232	.0683	.2500	.0683	-.0683	-.0378	-.0218	-.0420	-.0467	-.0420	-.0338	-.0259	-.0378	-.0538
4	.0101	.0232	.0683	.2500	-.0683	-.0232	-.0101	-.0468	-.0420	-.0338	-.0259	-.0259	-.0338	-.0420
5	-.0054	-.0101	-.0232	-.0683	.2500	.0683	.0232	-.0338	-.0378	-.0259	-.0180	-.0293	-.0339	-.0359
6	-.0136	-.0218	-.0378	-.0683	.0683	.2500	.0683	-.0683	-.0378	-.0218	-.0136	-.0339	-.0359	-.0339
7	-.0180	-.0259	-.0378	-.0538	.0232	.0683	.2500	-.0683	-.0232	-.0101	-.0054	-.0359	-.0339	-.0293
8	-.0259	-.0338	-.0420	-.0467	-.0101	-.0232	-.0683	.2500	.0683	.0232	.0101	-.0420	-.0338	-.0259
9	-.0338	-.0420	-.0467	-.0420	-.0218	-.0378	-.0683	.0683	.2500	.0683	.0232	-.0538	-.0378	-.0259
10	-.0420	-.0467	-.0420	-.0338	-.0259	-.0378	-.0538	.0232	.0683	.2500	.0683	-.0683	-.0378	-.0218
11	-.0467	-.0420	-.0338	-.0259	-.0259	-.0338	-.0420	.0101	.0232	.0683	.2500	-.0683	-.0232	-.0101
12	-.0538	-.0378	-.0259	-.0180	-.0293	-.0339	-.0359	-.0054	-.0101	-.0232	-.0683	.2500	.0683	.0232
13	-.0683	-.0378	-.0218	-.0136	-.0339	-.0359	-.0339	-.0136	-.0218	-.0378	-.0683	.0683	.2500	.0683
14	-.0683	-.0232	-.0101	-.0054	-.0359	-.0339	-.0293	-.0180	-.0259	-.0378	-.0538	.0232	.0683	.2500

TABLE III.- APPROXIMATE VALUES OF M FOR DIRICHLET PROBLEM

M	Iteration number										
	0	1	2	3	4	5	6	7	8	9	10
1	1.270	1.688	1.767	1.762	1.743	1.727	1.717	1.711	1.708	1.706	1.705
2	2.170	2.586	2.613	2.579	2.547	2.526	2.513	2.505	2.501	2.499	2.497
3	-1.970	-2.541	-2.775	-2.884	-2.939	-2.967	-2.982	-2.991	-2.995	-2.997	-2.999
4	-1.520	-2.182	-2.469	-2.592	-2.647	-2.674	-2.687	-2.694	-2.698	-2.770	-2.701
5	3.360	4.269	4.508	4.574	4.592	4.596	4.598	4.598	4.598	4.597	4.597
6	.840	1.083	1.156	1.183	1.193	1.198	1.200	1.200	1.201	1.201	1.201
7	-4.100	-4.925	-5.086	-5.109	-5.108	-5.104	-5.101	-5.100	-5.099	-5.098	-5.098
8	1.170	1.961	2.344	2.525	2.611	2.653	2.673	2.684	2.689	2.692	2.694
9	1.580	2.409	2.805	2.999	3.096	3.145	3.171	3.184	3.190	3.194	3.196
10	.900	1.393	1.663	1.815	1.899	1.944	1.968	1.980	1.987	1.991	1.993
11	-1.220	-1.292	-1.204	-1.122	-1.067	-1.036	-1.018	-1.008	-1.003	-1.001	-.999
12	-1.280	-1.669	-1.844	-1.927	-1.967	-1.986	-1.995	-2.000	-2.001	-2.002	-2.003
13	-.240	-.621	-.865	-.991	-1.052	-1.081	-1.095	-1.101	-1.104	-1.106	-1.106
14	-1.790	-2.452	-2.748	-2.884	-2.946	-2.975	-2.989	-2.995	-2.998	-3.000	-3.001

TABLE IV.- ERROR OF APPROXIMATE SOLUTION FOR DIRICHLET PROBLEM[a]

ϵ	Iteration number									
	1	2	3	4	5	6	7	8	9	10
1	0.418	0.080	-0.006	-0.019	-0.015	-0.010	-0.006	-0.003	-0.002	-0.001
2	.416	.028	-.034	-.032	-.021	-.013	-.007	-.004	-.002	-.001
3	-.571	-.233	-.109	-.055	-.029	-.015	-.008	-.004	-.002	-.001
4	-.662	-.286	-.123	-.056	-.027	-.013	-.007	-.004	-.002	-.001
5	.909	.239	.065	.018	.005	.001	0	0	0	0
6	.243	.073	.027	.011	.004	.002	.001	0	0	0
7	-.825	-.161	-.023	.002	.004	.003	.001	.001	0	0
8	.791	.383	.180	.086	.042	.021	.011	.005	.003	.001
9	.829	.396	.194	.097	.049	.025	.013	.007	.004	.002
10	.493	.270	.152	.084	.045	.024	.013	.007	.004	.002
11	-.072	.088	.082	.054	.032	.018	.010	.005	.003	.001
12	-.389	-.175	-.083	-.040	-.019	-.009	-.004	-.002	-.001	0
13	-.381	-.244	-.126	-.061	-.029	-.014	-.006	-.003	-.001	-.001
14	-.662	-.297	-.135	-.062	-.029	-.014	-.006	-.003	-.001	-.001

[a] $\epsilon^{(k+1)}$ is error of kth approximation.

TABLE V.- MATRICES FOR NEUMANN PROBLEM

(a) η_t' and η_t'' matrix

End points of exterior normal segments	Exterior normal segments																													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
1	-1	-1	1	-1	1	-1	1	-1	-1	-1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	-1	-1	-1	1	-1	1	-1	-1
2			-1		-1		-1					-1		-1					-1		-1		-1					-1		-1	1
3																																
4																																
5																																
6																																
7																																
8																																
9																																
10																																
11																																
12																																
13																																
14																																
15	1																															
16		1																														
17				1																												
18						1																										
19								1																								
20									1	1																						
21											1																					
22													1																			
23															1																	
24																1	1															
25																		1														
26																				1												
27																						1										
28																								1								
29																									1	1						
30																											1					
31																													1			
32																																

η_t''

TABLE V.- MATRICES FOR NEUMANN PROBLEM - Continued

(b) $(\eta_t''\eta')_t$ matrix

End points of exterior normal segments	Boundary points													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	3	-1												
2	-1	3	-1											
3		-1	3	-1										
4			-1	3										
5					3	-1								
6					-1	3	-1							
7						-1	3							
8								3	-1					
9								-1	3	-1				
10									-1	3	-1			
11										-1	3			
12												3	-1	
13												-1	3	-1
14													-1	3
15	-1													-1
16	-1													
17		-1												
18			-1											
19				-1										
20				-1	-1									
21					-1									
22						-1								
23							-1							
24							-1	-1						
25								-1						
26									-1					
27										-1				
28											-1			
29											-1	-1		
30												-1		
31													-1	
32														-1

TABLE V.- MATRICES FOR NEUMANN PROBLEM - Continued

(c) a' matrix

Row	Column													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0,0	1,0	2,0	3,0	4,1	4,2	4,3	3,4	2,4	1,4	0,4	1,3	1,2	1,1
2	1,0	0,0	1,0	2,0	3,1	3,2	3,3	2,4	1,4	0,4	1,4	2,3	2,2	2,1
3	2,0	1,0	0,0	1,0	2,1	2,2	2,3	1,4	0,4	1,4	2,4	3,3	3,2	3,1
4	3,0	2,0	1,0	0,0	1,1	1,2	1,3	0,4	1,4	2,4	3,4	4,3	4,2	4,1
5	4,1	3,1	2,1	1,1	0,0	0,1	0,2	1,3	2,3	3,3	4,3	5,2	5,1	5,0
6	4,2	3,2	2,2	1,2	0,1	0,0	0,1	1,2	2,2	3,2	4,2	5,1	5,0	5,1
7	4,3	3,3	2,3	1,3	0,2	0,1	0,0	1,1	2,1	3,1	4,1	5,0	5,1	5,2
8	3,4	2,4	1,4	0,4	1,3	1,2	1,1	0,0	1,0	2,0	3,0	4,1	4,2	4,3
9	2,4	1,4	0,4	1,4	2,3	2,2	2,1	1,0	0,0	1,0	2,0	3,1	3,2	3,3
10	1,4	0,4	1,4	2,4	3,3	3,2	3,1	2,0	1,0	0,0	1,0	2,1	2,2	2,3
11	0,4	1,4	2,4	3,4	4,3	4,2	4,1	3,0	2,0	1,0	0,0	1,1	1,2	1,3
12	1,3	2,3	3,3	4,3	5,2	5,1	5,0	4,1	3,1	2,1	1,1	0,0	0,1	0,2
13	1,2	2,2	3,2	4,2	5,1	5,0	5,1	4,2	3,2	2,2	1,2	0,1	0,0	0,1
14	1,1	2,1	3,1	4,1	5,0	5,1	5,2	4,3	3,3	2,3	1,3	0,2	0,1	0,0
15	1,0	2,0	3,0	4,0	5,1	5,2	5,3	4,4	3,4	2,4	1,4	0,3	0,2	0,1
16	0,1	1,1	2,1	3,1	4,2	4,3	4,4	3,5	2,5	1,5	0,5	1,4	1,3	1,2
17	1,1	0,1	1,1	2,1	3,2	3,3	3,4	2,5	1,5	0,5	1,5	2,4	2,3	2,2
18	2,1	1,1	0,1	1,1	2,2	2,3	2,4	1,5	0,5	1,5	2,5	3,4	3,3	3,2
19	3,1	2,1	1,1	0,1	1,2	1,3	1,4	0,5	1,5	2,5	3,5	4,4	4,3	4,2
20	4,0	3,0	2,0	1,0	0,1	0,2	0,3	1,4	2,4	3,4	4,4	5,3	5,2	5,1
21	5,1	4,1	3,1	2,1	1,0	1,1	1,2	2,3	3,3	4,3	5,3	6,2	6,1	6,0
22	5,2	4,2	3,2	2,2	1,1	1,0	1,1	2,2	3,2	4,2	5,2	6,1	6,0	6,1
23	5,3	4,3	3,3	2,3	1,2	1,1	1,0	2,1	3,1	4,1	5,1	6,0	6,1	6,2
24	4,4	3,4	2,4	1,4	0,3	0,2	0,1	1,0	2,0	3,0	4,0	5,1	5,2	5,3
25	3,5	2,5	1,5	0,5	1,4	1,3	1,2	0,1	1,1	2,1	3,1	4,2	4,3	4,4
26	2,5	1,5	0,5	1,5	2,4	2,3	2,2	1,1	0,1	1,1	2,1	3,2	3,3	3,4
27	1,5	0,5	1,5	2,5	3,4	3,3	3,2	2,1	1,1	0,1	1,1	2,2	2,3	2,4
28	0,5	1,5	2,5	3,5	4,4	4,3	4,2	3,1	2,1	1,1	0,1	1,2	1,3	1,4
29	1,4	2,4	3,4	4,4	5,3	5,2	5,1	4,0	3,0	2,0	1,0	0,1	0,2	0,3
30	2,3	3,3	4,3	5,3	6,2	6,1	6,0	5,1	4,1	3,1	2,1	1,0	1,1	1,2
31	2,2	3,2	4,2	5,2	6,1	6,0	6,1	5,2	4,2	3,2	2,2	1,1	1,0	1,1
32	2,1	3,1	4,1	5,1	6,0	6,1	6,2	5,3	4,3	3,3	2,3	1,2	1,1	1,0

TABLE V.- MATRICES FOR NEUMANN PROBLEM - Continued

(d) A' matrix

Row	Column													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	0.2500	0.3634	0.4303	0.4824	0.4960	0.5140	0.5140	0.4960	0.4824	0.4770	0.4404	0.3866	0.3183
2	.2500	0	.2500	.3634	.4404	.4622	.4881	.4960	.4824	.4770	.4824	.4622	.4244	.3866
3	.3634	.2500	0	.2500	.3866	.4244	.4622	.4824	.4770	.4824	.4960	.4881	.4622	.4404
4	.4303	.3634	.2500	0	.3183	.3866	.4404	.4770	.4824	.4960	.5140	.5140	.4960	.4824
5	.4824	.4404	.3866	.3183	0	.2500	.3634	.4404	.4622	.4881	.5140	.5253	.5163	.5129
6	.4960	.4622	.4244	.3866	.2500	0	.2500	.3866	.4244	.4622	.4960	.5163	.5129	.5163
7	.5140	.4881	.4622	.4404	.3634	.2500	0	.3183	.3866	.4404	.4824	.5129	.5163	.5253
8	.5140	.4960	.4824	.4770	.4404	.3866	.3183	0	.2500	.3634	.4303	.4824	.4960	.5140
9	.4960	.4824	.4770	.4824	.4622	.4244	.3866	.2500	0	.2500	.3634	.4404	.4622	.4881
10	.4824	.4770	.4824	.4960	.4881	.4622	.4404	.3634	.2500	0	.2500	.3866	.4244	.4622
11	.4770	.4824	.4960	.5140	.5140	.4960	.4824	.4303	.3634	.2500	0	.3183	.3866	.4404
12	.4404	.4622	.4881	.5140	.5253	.5163	.5129	.4824	.4404	.3866	.3183	0	.2500	.3634
13	.3866	.4244	.4622	.4960	.5163	.5129	.5163	.4960	.4622	.4244	.3866	.2500	0	.2500
14	.3183	.3866	.4404	.4824	.5129	.5163	.5253	.5140	.4881	.4622	.4404	.3634	.2500	0
15	.2500	.3634	.4303	.4770	.5163	.5253	.5382	.5336	.5140	.4960	.4824	.4303	.3634	.2500
16	.2500	.3183	.3866	.4404	.4960	.5140	.5336	.5382	.5253	.5163	.5129	.4824	.4404	.3866
17	.3183	.2500	.3183	.3866	.4622	.4881	.5140	.5253	.5163	.5129	.5163	.4960	.4622	.4244
18	.3866	.3183	.2500	.3183	.4244	.4622	.4960	.5163	.5129	.5163	.5253	.5140	.4881	.4622
19	.4404	.3866	.3183	.2500	.3866	.4404	.4824	.5129	.5163	.5253	.5382	.5336	.5140	.4960
20	.4770	.4303	.3634	.2500	.2500	.3634	.4303	.4824	.4960	.5140	.5336	.5382	.5253	.5163
21	.5163	.4824	.4404	.3866	.2500	.3183	.3866	.4622	.4881	.5140	.5382	.5508	.5444	.5421
22	.5253	.4960	.4622	.4244	.3183	.2500	.3183	.4244	.4622	.4960	.5253	.5444	.5421	.5444
23	.5382	.5140	.4881	.4622	.3866	.3183	.2500	.3866	.4404	.4824	.5163	.5421	.5444	.5508
24	.5336	.5140	.4960	.4824	.4303	.3634	.2500	.2500	.3634	.4303	.4770	.5163	.5253	.5382
25	.5382	.5253	.5163	.5129	.4824	.4404	.3866	.2500	.3183	.3866	.4404	.4960	.5140	.5336
26	.5253	.5163	.5129	.5163	.4960	.4622	.4244	.3183	.2500	.3183	.3866	.4622	.4881	.5140
27	.5163	.5129	.5163	.5253	.5140	.4881	.4622	.3866	.3183	.2500	.3183	.4244	.4622	.4960
28	.5129	.5163	.5253	.5382	.5336	.5140	.4960	.4404	.3866	.3183	.2500	.3866	.4404	.4824
29	.4824	.4960	.5140	.5336	.5382	.5253	.5163	.4770	.4303	.3634	.2500	.2500	.3634	.4303
30	.4622	.4881	.5140	.5382	.5508	.5444	.5421	.5163	.4824	.4404	.3866	.2500	.3183	.3866
31	.4244	.4622	.4960	.5253	.5444	.5421	.5444	.5253	.4960	.4622	.4244	.3183	.2500	.3183
32	.3866	.4404	.4824	.5163	.5421	.5444	.5508	.5382	.5140	.4881	.4622	.3866	.3183	.2500

TABLE V.- MATRICES FOR NEUMANN PROBLEM - Concluded

(e) C' matrix

Row	Column													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0.7500	-0.0683	-0.0233	-0.0101	0.0055	0.0135	0.0179	0.0258	0.0337	0.0421	0.0467	0.0537	0.0684	0.0683
2	-.0683	.7500	-.0683	-.0233	.0100	.0219	.0259	.0337	.0421	.0467	.0421	.0379	.0378	.0233
3	-.0233	-.0683	.7500	-.0683	.0233	.0378	.0379	.0421	.0467	.0421	.0337	.0259	.0219	.0100
4	-.0101	-.0233	-.0683	.7500	.0683	.0684	.0537	.0467	.0421	.0337	.0258	.0179	.0135	.0055
5	.0421	.0537	.0684	.0683	.7500	-.0683	-.0233	.0100	.0219	.0258	.0258	.0294	.0337	.0360
6	.0337	.0379	.0378	.0233	-.0683	.7500	-.0683	.0233	.0378	.0379	.0337	.0337	.0360	.0337
7	.0258	.0259	.0219	.0100	-.0233	-.0683	.7500	.0683	.0684	.0537	.0421	.0360	.0337	.0294
8	.0258	.0337	.0421	.0467	.0537	.0684	.0683	.7500	-.0683	-.0233	-.0101	.0055	.0135	.0179
9	.0337	.0421	.0467	.0421	.0379	.0378	.0237	-.0683	.7500	-.0683	-.0233	.0100	.0219	.0259
10	.0421	.0467	.0421	.0337	.0259	.0219	.0100	-.0233	-.0683	.7500	-.0683	.0233	.0378	.0379
11	.0467	.0421	.0337	.0258	.0179	.0135	.0055	-.0101	-.0233	-.0683	.7500	.0683	.0684	.0537
12	.0100	.0219	.0259	.0258	.0294	.0337	.0360	.0421	.0537	.0684	.0683	.7500	-.0683	-.0233
13	.0233	.0378	.0379	.0337	.0337	.0360	.0337	.0337	.0379	.0378	.0233	-.0683	.7500	-.0683
14	.0683	.0684	.0537	.0421	.0360	.0337	.0294	.0258	.0259	.0219	.0100	-.0233	-.0683	.7500

TABLE VI.- APPROXIMATE VALUES OF m FOR NEUMANN PROBLEM

[Given values of $\Delta_1\bar{u}$ taken as $m^{(0)}$ are -0.60, -0.46, 0.32, -1.06, 1.48, 0.65, -0.03, -0.64, -0.42, -0.77, -0.78, 1.06, 0.30, and 0.95]

m	Iteration number													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	-0.956	-1.163	-1.278	-1.338	-1.363	-1.367	-1.359	-1.341	-1.319	-1.293	-1.265	-1.235	-1.205	-1.175
2	-.767	-.979	-1.129	-1.241	-1.327	-1.395	-1.449	-1.493	-1.529	-1.558	-1.581	-1.599	-1.613	-1.623
3	.674	1.020	1.338	1.621	1.866	2.077	2.257	2.409	2.539	2.649	2.743	2.824	2.892	2.952
4	-1.178	-2.286	-2.647	-2.910	-3.104	-3.247	-3.353	-3.431	-3.487	-3.526	-3.552	-3.567	-3.574	-3.575
5	2.466	3.148	3.637	3.999	4.273	4.485	4.653	4.788	4.898	4.989	5.066	5.132	5.189	5.240
6	.980	1.152	1.246	1.301	1.338	1.366	1.390	1.412	1.432	1.452	1.471	1.489	1.507	1.525
7	-.233	-.473	-.696	-.881	-1.027	-1.136	-1.215	-1.270	-1.305	-1.325	-1.334	-1.334	-1.328	-1.317
8	-.984	-1.179	-1.297	-1.374	-1.426	-1.461	-1.486	-1.501	-1.509	-1.511	-1.508	-1.501	-1.491	-1.477
9	-.568	-.587	-.548	-.486	-.416	-.348	-.283	-.224	-.170	-.122	-.078	-.038	-.002	-.031
10	-1.195	-1.435	-1.573	-1.655	-1.706	-1.737	-1.758	-1.771	-1.779	-1.783	-1.785	-1.785	-1.783	-1.780
11	-1.181	-1.373	-1.451	-1.466	-1.448	-1.414	-1.371	-1.327	-1.282	-1.238	-1.196	-1.156	-1.118	-1.082
12	1.863	2.478	2.955	3.327	3.621	3.855	4.043	4.194	4.317	4.418	4.500	4.569	4.626	4.674
13	.320	2.052	.034	-.149	-.325	-.483	-.619	-.733	-.828	-.904	-.965	-1.013	-1.050	-1.077
14	1.537	1.920	2.178	2.364	2.507	2.623	2.722	2.808	2.886	2.956	3.021	3.081	3.136	3.188

TABLE VII.- ERROR OF APPROXIMATE SOLUTION FOR NEUMANN PROBLEM

ϵ (a)	Iteration number													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	-0.356	-0.207	-0.115	-0.060	-0.025	-0.004	0.009	0.017	0.023	0.026	0.028	0.029	0.030	0.030
2	-.307	-.211	-.151	-.112	-.086	-.068	-.054	-.044	-.036	-.289	-.231	-.018	-.014	-.010
3	.354	.346	.318	.282	.245	.211	.180	.153	.130	.110	.094	.080	.069	.059
4	-.722	-.505	-.361	-.263	-.194	-.143	-.106	-.078	-.056	-.039	-.026	-.015	-.007	-.001
5	.986	.682	.489	.362	.274	.212	.168	.135	.110	.091	.077	.066	.057	.051
6	.330	.172	.094	.055	.037	.028	.024	.022	.020	.020	.019	.018	.018	.017
7	-.203	-.241	-.223	-.185	-.145	-.109	-.079	-.055	-.035	-.020	-.009	0	.006	.011
8	-.344	-.195	-.118	-.077	-.052	.036	-.024	-.015	-.008	-.002	.003	.007	.011	.014
9	-.148	-.019	.039	.063	.069	.069	.064	.059	.054	.049	.044	.040	.036	.033
10	-.425	-.240	-.138	-.082	-.050	-.032	-.020	-.013	-.008	-.005	-.002	0	.002	.003
11	-.401	-.192	-.077	-.015	.018	.035	.042	.045	.045	.044	.042	.040	.038	.036
12	.803	.615	.476	.372	.294	.234	.188	.151	.123	.101	.083	.068	.057	.048
13	.020	-.115	-.171	-.184	-.176	-.158	-.136	-.115	-.094	-.077	-.061	-.048	-.036	-.027
14	.590	.380	.258	.186	.143	.116	.099	.087	.078	.071	.065	.060	.056	.052

[a] $\epsilon^{(k+1)}$ is error of kth approximation.

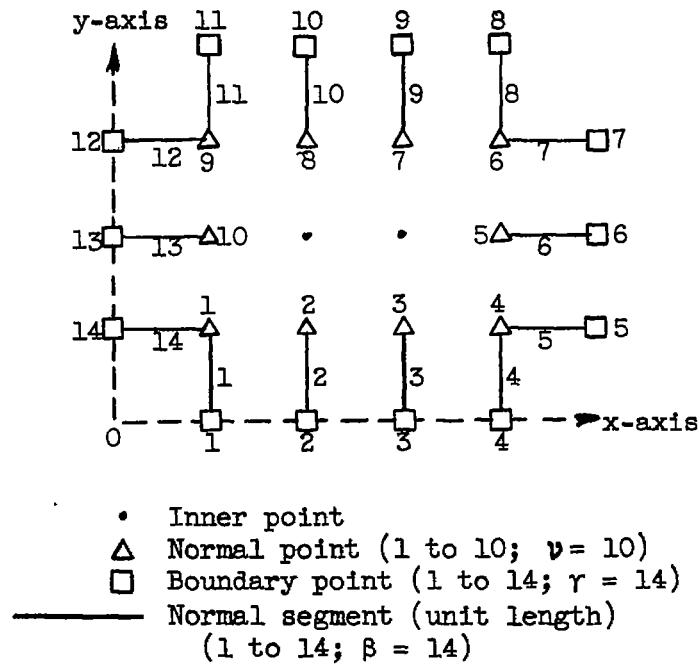

Figure 1.- Region showing boundary points, normal points, and inner normal segments used in calculation of C for Dirichlet problem.

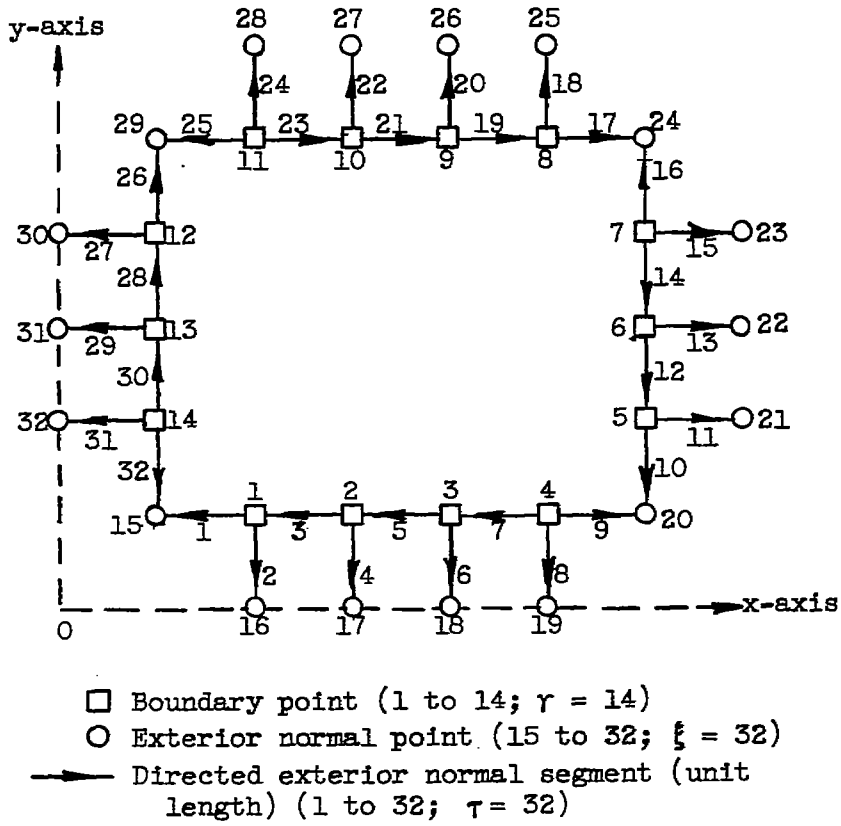

Figure 2.- Region showing boundary points, exterior normal points, and exterior normal segments used in calculation of C' for Neumann problem.

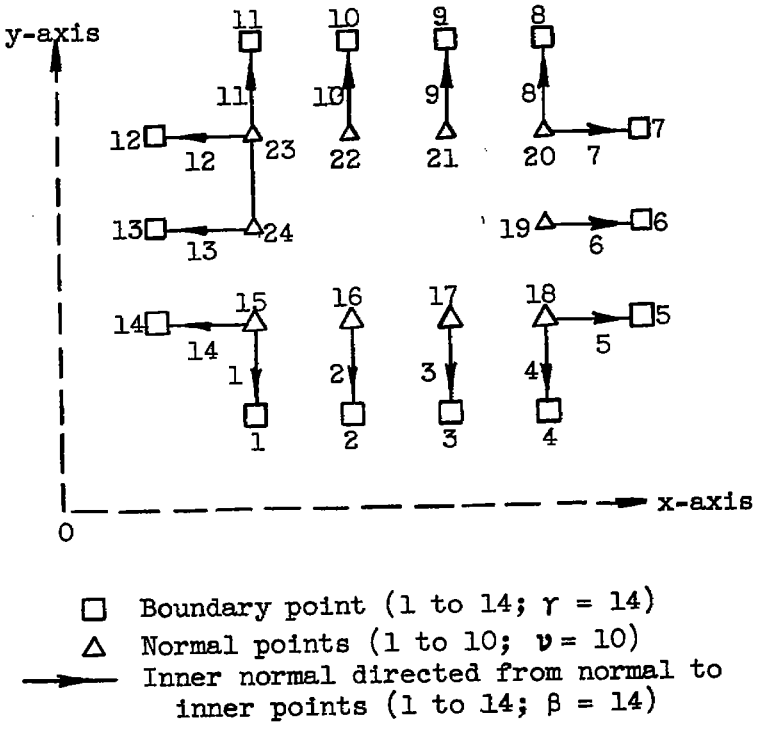

Figure 3.- Region showing boundary points, inner normal points, and inner segments used in calculation of C'' in Dirichlet problem.

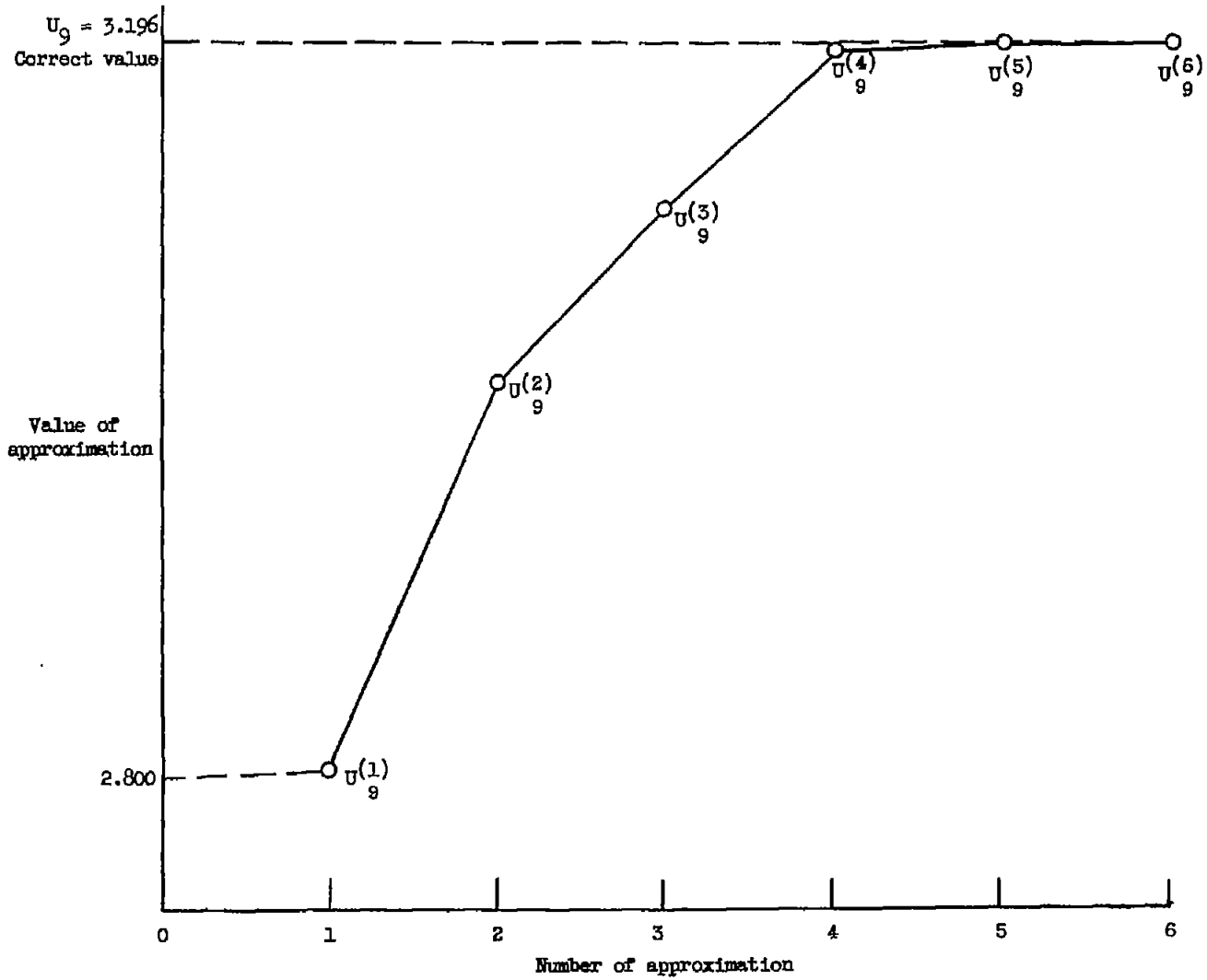

Figure 4.- Error of successive approximations at ninth boundary point. (For approximations 2 to 13 this is bound for absolute error.)

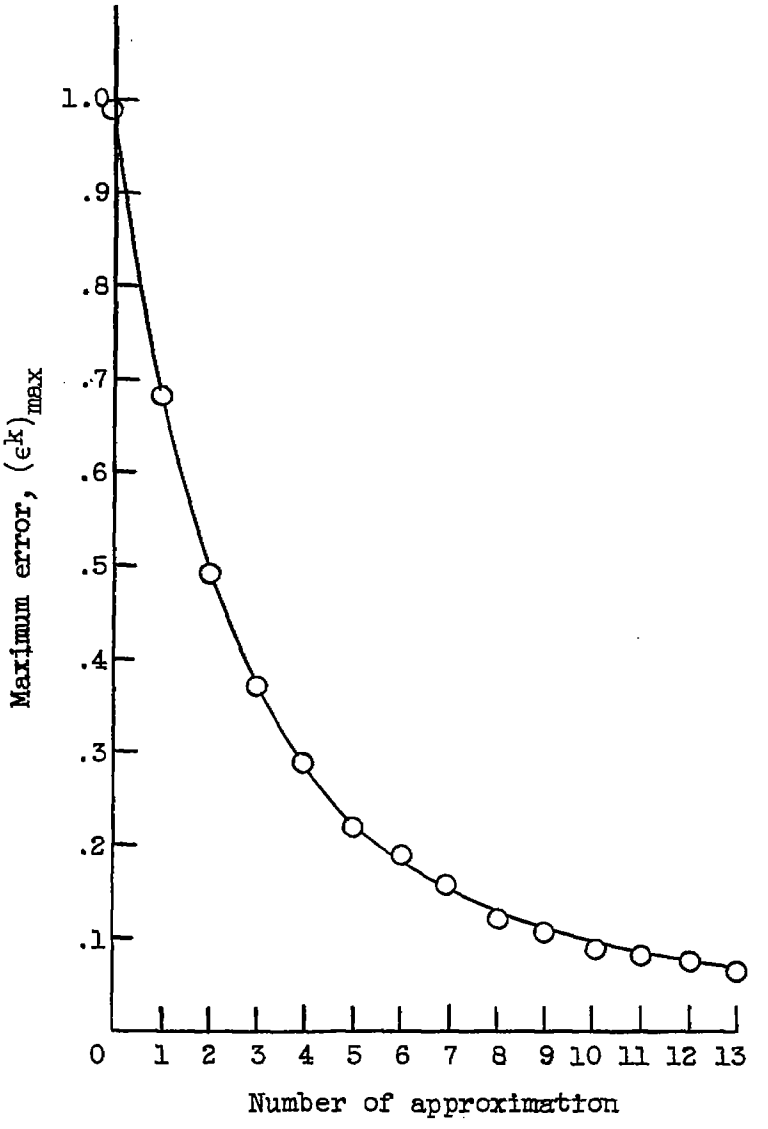

Figure 5.- Maximum error $\left[\epsilon^{(k)}\right]_{max}$ at boundary for Neuman problem where $\left[\epsilon^{(k)}\right]_{max} = \max_{1 \leq r \leq 14} \left|\epsilon_r^{(k+1)}\right|$ and $\epsilon_r^{(k+1)} = \Delta_1 U_r - \Delta_1 U_r^{(k)}$.